On Equivalences Between Module Subcategories

by

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The undersigned certify that we have read a thesis, entitled ``On equivalences Between Module Subcategories" submitted to the Graduate School by Leung Chi Kwan in partial fulfillment of the requirements of the degree of Master of Philosophy in Mathematics. We recommend that it be accepted.

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Abstract

Let A and R be rings with unit , $\mathcal{C}_A$ and $\mathcal{G}_R$ be full subcategories of $Mod\text{-}A$ and $Mod\text{-}R$ respectively . This thesis is intended to be an up to date account of the major results in the theory of equivalences between module subcategories $\mathcal{C}_A$ and $\mathcal{G}_R$. We concentrate on the representable equivalences .

The classical Morita equivalence theorem characterizes the category equivalences between $Mod\text{-}A$ and $Mod\text{-}R$ by *progenerators* . Later , Fuller generalized Morita's theorem to the equivalences between $Mod\text{-}A$ and a subcategory $\mathcal{G}_R(\subseteq Mod\text{-}R)$ which is closed under direct sums , quotients and submodules and he characterized such equivalences by *quasi-progenerators* . We prove Fuller's result via Azumaya's approach .

More recently , Menini and Orsatti further generalized Fuller's context and showed that every abstract equivalence between the category $\mathcal{C}_A \subseteq Mod\text{-}A$ which is closed under taking submodules and such that $A_A \in \mathcal{C}_A$ and the category $\mathcal{G}_R \subseteq Mod\text{-}R$ which is closed under taking epimorphic images and arbitrary direct sums is represented by a bimodule ${}_AP_R$, with $A \cong End(P_R)$, and the representing module is uniquely determined by the equivalence and is then called a $*$-module .

There is a very important class of modules known as tilting modules which first come from the representation theory of finite dimensional algebras . Menini and Orsatti proved that every tilting module is a $*$-module and then provided examples of tilting modules which are not quasi-progenerators and so their result is a non-trivial generalization of Fuller's theorem . Later , Colpi characterized tilting modules as the representing modules of equivalences between certain module subcategories .

Until most recently , Trlifaj proved that every $*$-module over an arbitrary ring is finitely generated . Also the relations between $*$-modules , tilting modules , quasiprogenerators and progenerators are studied by Colpi , Menini and Trlifaj .

Some module equivalences can be characterized by dualities . We present the equivalence $\mathcal{P}_A \sim \mathcal{I}_R$ studied by Colpi and the equivalence $FGP\text{-}A \sim FCI\text{-}R$ studied by Xue .

Finally , as applications of the previous theory , we give the Tilting Theorem due to Colby and Fuller that characterizes the tilting modules . Also some conditions due to P. A. Guil Asensio and F. Guil Asensio that solve the problem of when two tilting modules define the same torsion theories are presented . And then we record some results that characterize the isomorphisms of the endomorphism rings of the above representing modules by category equivalences .

Contents

Preface

Let A and R be rings with unit , $\mathcal{C}_A$ and $\mathcal{G}_R$ be full subcategories of $Mod\text{-}A$ and $Mod\text{-}R$ respectively . This thesis is a survey of the main results in the theory of equivalences between module subcategories $\mathcal{C}_A$ and $\mathcal{G}_R$. In fact , we focus on the equivalences between module subcategories that can be represented by a bimodule ${}_AP_R$ via the pair of functors $T_P = (- \otimes_A P)$ and $H_P = Hom_R(P, -)$.

Chapter 1 gives an introduction to the theory of equivalences between module categories . It contains some notations , basic definitions and lemmas that are useful in the following chapters .

Chapter 2 includes some classical results of the theory of equivalence between module categories . The classical Morita equivalence theorem states that the additive functors $F : Mod\text{-}A \rightleftarrows Mod\text{-}R : G$ define an category equivalence if and only if there exists a *progenerator* P_R with $A \cong End(P_R)$ canonically . Moreover $F \cong (- \otimes_A P)$ and $G \cong Hom_R(P, -)$. The reader is referred to [AF] for a complete proof of Morita's theorem .

Later , as a generalization of Morita's result , Fuller [F , 1974] considered the equivalences between $Mod\text{-}A$ and a subcategory $\mathcal{G}_R(\subseteq Mod\text{-}R)$ which is closed under direct sums , quotients and submodules and he proved that every equivalence between $Mod\text{-}A$ and $\mathcal{G}_R$ is represented by a *quasi-progenerator* P_R , with $A \cong End(P_R)$ canonically , via the pair of functors $T_P = (- \otimes_A P)$ and $H_P = Hom_R(P, -)$ and $\mathcal{G}_R = Gen(P_R)$. Conversely if P_R is a quasi-progenerator with $A = End(P_R)$ then it induces such an equivalence . We prove Fuller's result via Azumaya's approach [A , 1979] .

A general technique in studying equivalences between module categories is that for a bimodule ${}_AP_R$, we set $T_P = (- \otimes_A P)$ and $H_P = Hom_R(P, -)$

and suppose that

$$\mathcal{C}_A \underset{H_P}{\overset{T_P}{\rightleftarrows}} \mathcal{G}_R$$

defines an equivalence , then we try to obtain some characterizations of the bimodule ${}_AP_R$ and properties of the equivalence . In this way , M. Sato ([S1] and [S2]) obtained some useful results for the equivalences : $Mod\text{-}A \sim Im(T_P)$, $Mod\text{-}A \sim Gen(P_R)$, $Im(H_P) \sim Im(T_P)$ and $Im(H_P) \sim \overline{Gen}(P_R)$.

Naturally , in view of Fuller's result , one may try to weaken the conditions and find equivalences between module subcategories that are still represented by some modules .

Chapter 3 presents a further generalization of Fuller's context . Recently , Menini and Orsatti [MO , 1989] proved that if there are given subcategories : $\mathcal{C}_A \subseteq Mod\text{-}A$ which is closed under taking submodules and such that $A_A \in \mathcal{C}_A$ and $\mathcal{G}_R \subseteq Mod\text{-}R$ which is closed under taking epimorphic images and arbitrary direct sums , then every abstract equivalence between them is represented by a bimodule ${}_AP_R$, with $A \cong End(P_R)$ canonically , via the pair of functors $T_P = (- \otimes_A P)$ and $H_P = Hom_R(P, -)$ and $\mathcal{C}_A = Cogen(Hom_R(P, Q_R))$ for a cogenerator Q_R and $\mathcal{G}_R = Gen(P_R)$. Moreover the representing module is uniquely determined by the equivalence and is then called a $*$-module .

There is a very important class of modules in contemporary module theory known as tilting modules which first come from the representation theory of finite dimensional algebras . Menini and Orsatti [MO] proved that every tilting module is a $*$-module and then provided examples of tilting modules which are NOT quasi-progenerators and so their result is a non-trivial generalization of Fuller's theorem . Later , Colpi [C2 , 1993] characterized tilting modules as the representing modules of equivalences between certain module subcategories .

Until most recently , Trlifaj [T2 , 1994] proved that every $*$-module over an arbitrary ring is finitely generated . This is the most important result about the structure of the $*$-modules .

Also the relations between $*$-modules , tilting modules , quasiprogenerators and progenerators are studied by Colpi , Menini ([CM , 1993] and [C2 , 1993]) and Trlifaj [T2 , 1994] .

Chapter 4 contains some interesting module equivalences that can be characterized by dualities . Iet R be a ring , we denote with R-*mod* (*mod*-R) the category of all finitely generated left (right) R-modules , $\mathcal{P}_R$ ($\mathcal{I}_R$) the category of all projective (injective) right R-modules and FGP-R (FCI-R) the category of all finitely generated projective (finitely cogenerated injective) right R-modules . Similarly we introduce the notations ${}_R\mathcal{P}$, ${}_R\mathcal{I}$, R-FGP and R-FCI for left R-modules .

R. Colpi [C3 , 1993] proved that for a bimodule ${}_AU_R$,

$$Hom_A(-, {}_AU_R) : A\text{-}Mod \rightleftarrows Mod\text{-}R : Hom_R(-, {}_AU_R)$$

induce a duality between A-*mod* and *mod*-R if and only if there exists an equivalence between $\mathcal{P}_A$ and $\mathcal{I}_R$. Moreover , the equivalence is induced by the pair of functors $T_U = (- \otimes_A U)$ and $H_U = Hom_R(U, -))$.

Also Xue [X , 1995] obtained that there is a duality between A-FGP and FCI-R if and only if there is an equivalence between FGP-A and FCI-R . Again , in this case , the equivalence is induced by the pair of functors $T_U = (- \otimes_A U)$ and $H_U = Hom_R(U, -))$.

As applications of the previous theory , in the last chapter , we give a general version of Tilting Theorem due to Colby and Fuller [CbF1 , 1990] that actually characterizes the tilting modules .

Moreover , some conditions due to P. A. Guil Asensio and F. Guil Asensio [GG , 1992] that solve the problem of when two tilting modules induce the same torsion theories are presented .

Finally we record some results that characterize the isomorphisms of the endomorphism rings of the representing modules by category equivalences . Bolla [B , 1984] showed that if there are given progenerators P_R and Q_S , then every ring isomorphism δ : $End(P_R) \longrightarrow End(Q_S)$ is induced by a category equivalence F_δ : Mod-$R \longrightarrow Mod$-S which is unique up to natural isomorphism . Later Lok and Shum [LS , 1995] generalized Bolla's result to the class of quasi-progenerators . And P.A. Guil Asensio and F. Guil Asensio [GG] obtained another generalization to the class of tilting modules . However , in general , the class of quasi-progenerators $\neq$ the class of tilting modules .

Chapter 1

Introduction to Module Equivalence

1.1 Introduction and Preliminaries

In the following , every ring has a nonzero identity and is associative but not necessary commutative . And over a ring , every module is unital . For every ring R , *Mod-R* (*R-Mod*) denotes the category of all right (left) R-modules . And we use the symbol M_R ($_RM$) to emphasize that M is a right (left) R-module .

Categories and functors are always additive . Every subcategory of a given category is full and closed under isomorphic objects . Unless noted otherwise , we will follow the notation and conventions of [AF] . And our main reference books of ring theory are [AF] , [Fa] , [R] and [W] . Actually the reader should be familiar with the material in chapter 5-6 of [AF] .

Let A and R be rings , $\mathcal{C}_A$ and $\mathcal{G}_R$ be subcategories of *Mod-A* and *Mod-R* respectively . Our main interest is to study the category equivalences between $\mathcal{C}_A$ and $\mathcal{G}_R$. We use " $\mathcal{C}_A \sim \mathcal{G}_R$ " to denote that $\mathcal{C}_A$ and $\mathcal{G}_R$ are equivalent .

We say that a category equivalence

$$\mathcal{C}_A \overset{G}{\underset{F}{\rightleftarrows}} \mathcal{G}_R$$

is defined (here G , F are additive functors) if there exists natural isomorphisms $\mu : GF \to Id_{\mathcal{G}_R}$ and $\nu : Id_{\mathcal{C}_A} \to FG$ i.e. we have the commutative diagrams with isomorphic columns :

$$\begin{array}{ccc} GF(M) & \xrightarrow{GFf} & GF(M') \\ {\scriptstyle \mu_M}\downarrow & & \downarrow{\scriptstyle \mu_{M'}} \\ M & \xrightarrow{f} & M' \end{array} \qquad \begin{array}{ccc} N & \xrightarrow{h} & N' \\ {\scriptstyle \nu_N}\downarrow & & \downarrow{\scriptstyle \nu_{N'}} \\ FG(N) & \xrightarrow{FGh} & FG(N') \end{array}$$

for any M , $M' \in \mathcal{G}_R$ and N , $N' \in \mathcal{C}_A$. And in this case , by [AF , 21.2 and 21.3] ,
(i) F and G are full and faithful functors i.e.

$$Hom_A(N, N') \overset{G}{\cong} Hom_R(GN, GN')$$

$$Hom_R(M, M') \overset{F}{\cong} Hom_A(FM, FM')$$

where N , $N' \in \mathcal{C}_A$ and M , $M' \in \mathcal{G}_R$. Note that here $\mathcal{C}_A$ and $\mathcal{G}_R$ are *full* subcategories and
(ii) (F, G) and (G, F) are adjoint pairs with respect to categories $\mathcal{C}_A$ and $\mathcal{G}_R$ i.e. we have the isomorphisms ,

$$\Phi_1(M, N) : Hom_A(N, FM) \cong Hom_R(GN, M)$$

which is natural in $N \in \mathcal{C}_A$, $M \in \mathcal{G}_R$ and

$$\Phi_2(L, K) : Hom_A(FL, K) \cong Hom_R(L, GK)$$

which is natural in $L \in \mathcal{G}_R$, $K \in \mathcal{C}_A$.

1.1.1 Let ${}_AP_R$ be a bimodule . Recall that we have the functors

$$T_P = (- \otimes_A P) : Mod\text{-}A \to Mod\text{-}R \ \text{ and}$$

$$H_P = Hom_R(P_R, -) : Mod\text{-}R \to Mod\text{-}A$$

and associated with these functors T_P and H_P , there are natural morphisms $\rho : T_P H_P \longrightarrow Id_{Mod\text{-}R}$ and $\sigma : Id_{Mod\text{-}A} \longrightarrow H_P T_P$ defined as follows :
for every $M \in Mod\text{-}R$,

$$\rho_M : Hom_R(P_R, M) \otimes_A P \to M \quad [f \otimes p \mapsto f(p)]$$

and for every $N \in Mod\text{-}A$,

$$\sigma_N : N \to Hom_R(P_R, N \otimes_A p) \quad [n \mapsto [p \mapsto n \otimes p]] \quad .$$

We call σ and ρ the natural morphisms associated with the functors T_P and H_P . Sometimes we may write $\rho(M)$ and $\sigma(N)$ for ρ_M and σ_N respectively .

An **equivalence** $G : \mathcal{C}_A \rightleftarrows \mathcal{G}_R : F$ is said to be *representable* if there exists a bimodule ${}_AP_R$ such that the following natural isomorphisms of functors hold :

$$F \cong Hom_R(P_R, -)|_{\mathcal{G}_R} = H_P|_{\mathcal{G}_R}$$
$$G \cong (- \otimes_A P)|_{\mathcal{C}_A} = T_P|_{\mathcal{C}_A} \quad .$$

In this case we say that the bimodule ${}_AP_R$ (or the pair of functors (T_P, H_P)) induces the equivalence (G, F) .

1.1.2 Adjoint Isomorphism . It is well known that (H_P, T_P) is an adjoint pair [AF 20.6] . That is for every $N_A \in Mod\text{-}A$ and $M_R \in Mod\text{-}R$, we have the natural isomorphism

$$\Phi : Hom_A(N, Hom_R(P, M)) \longrightarrow Hom_R(N \otimes_A P, M)$$

defined by $\Phi(g)[(n \otimes p)] = g(n)(p)$ for every $g \in Hom_A(N, Hom_R(P, M))$ and $(n \otimes p) \in N \otimes_A P$. Moreover

$$\Phi^{-1} : Hom_R(N \otimes_A P, M) \longrightarrow Hom_A(N, Hom_R(P, M))$$

is given by $[\Phi^{-1}(h)(n)](p) = h(n \otimes p)$ where $h \in Hom_R(N \otimes_A P, M)$, $n \in N$ and $p \in P$.

Note here that $\Phi(Id_{H(M)}) = \rho_M$ for every M_R and $\Phi^{-1}(Id_{T(N)}) = \sigma_N$ for every N_A .

Lemma 1.1.3 *Let $N_A \in Mod\text{-}A$, $M_R \in Mod\text{-}R$. For brevity , here we write $T = T_P$ and $H = H_P$. Then , with notations above , we have :*

(a) $\rho_{T(N)} \circ T(\sigma_N) = Id_{T(N)}$ and hence $\rho_{T(N)}$ is always epic . Moreover if σ_N is an isomorphism then $\rho_{T(N)}$ is an isomorphism .

(b) $H(\rho_M) \circ \sigma_{H(M)} = Id_{H(M)}$ and hence $\sigma_{H(M)}$ is always monic . Moreover if ρ_M is an isomorphism then $\sigma_{H(M)}$ is an isomorphism .

(c) $g = \rho_M \circ T_P(\Phi^{-1}(g))$ for each $g \in Hom_R(N \otimes_A P, M)$.

(d) $f = H_P(\Phi(f)) \circ \sigma_N$ for each $f \in Hom_A(N, Hom_R(P, M))$.

Proof :
(a)

$$\begin{array}{ccc} T(N) & & \\ {\scriptstyle T(\sigma_N)}\downarrow & & \\ THT(N) & \xrightarrow{\rho_{T(N)}} & T(N) \end{array}$$

Let $n \in N$ and $p \in P$ then we have

$$\begin{aligned} \rho_{T(N)} \circ T(\sigma_N)(n \otimes p) &= (\rho_{T(N)} \circ (\sigma_N \otimes Id_P))(n \otimes p) \\ &= \rho_{T(N)}(\sigma_N(n) \otimes p) \\ &= \sigma_N(n)(p) \\ &= n \otimes p \end{aligned}$$

. So $\rho_{T(N)} \circ T(\sigma_N) = Id_{T(N)}$.

(b)

$$\begin{array}{ccc} H(M) & & \\ {\scriptstyle \sigma_{H(M)}}\downarrow & & \\ HTH(M) & \xrightarrow{H(\rho_M)} & H(M) \end{array}$$

Let $f \in H_P(M) = Hom_R(P, M)$ and $p \in P$ then we have $[H(\rho_M) \circ \sigma_{H(M)}](f)(p) = \rho_M[\sigma_{H(M)}(f)(p)] = \rho_M(f \otimes p) = f(p)$.

(c) Let $n \in N$ and $p \in P$ then

$$\begin{aligned}(\rho_M \circ T_P(\Phi^{-1}(g)))(n \otimes p) &= (\rho_M \circ (\Phi^{-1}(g)) \otimes Id_P)(n \otimes p) \\ &= \rho_M(\Phi^{-1}(g)(n) \otimes p) \\ &= \Phi^{-1}(g)(n)(p) \\ &= g(n \otimes p)\end{aligned}$$

and so $g = \rho_M \circ T_P(\Phi^{-1}(g))$.

(d) $[(H_P(\Phi(f)) \circ \sigma_N)](n)(p) = [(\Phi(f)) \circ \sigma_N)](n)(p) = \Phi(f)(n \otimes p) = f(n)(p)$, for every $n \in N$ and $p \in P$. □

A bimodule ${}_AM_R$ is *faithfully balanced* if the canonical morphisms

$$A \to End(M_R) \;\; [a \to [m \to am]] \;\; \text{and} \;\; R \to End({}_AM) \;\; [r \to [m \to mr]]$$

are ring isomorphisms .

Let $P \in Mod\text{-}R$, a module $M \in Mod\text{-}R$ is *P-generated* if there exists a set X and an epimorphism $\phi : P^{(X)} \to M \to 0$. $Gen(P_R)$ denotes the category of all right P-generated R-modules , and $\overline{Gen}(P_R)$ denotes the smallest subcategory of $Mod\text{-}R$ containing Gen(P_R) and closed under taking submodules . Clearly $Gen(P_R) = \overline{Gen}(P_R)$ if and only if $Gen(P_R)$ is closed under submodules .

We set $t_P(M) = \{ \sum f(P) \mid f \in Hom_R(P, M)\}$ i.e. the trace of P in M and $t_P(M)$ is the largest submodule of M which belongs to $Gen(P_R)$.

Note that $Hom_R(P, M)$ is canonically isomorphic to $Hom_R(P, t_P(M))$ and so $\rho_{t_P(M)}$ is an isomorphism if and only if ρ_M is monic . It follows that ρ_M is an isomorphism for all $M \in Gen(P_R)$ if and only if ρ_M is monic for all $M \in Mod\text{-}R$.

We say that a module M_R is *P-presented* if there exists a *P-presentation* of M , i.e. there is an exact sequence : $P^{(X)} \xrightarrow{\psi} P^{(Y)} \xrightarrow{\varphi} M \to 0$, where X

and Y are sets . So M and $Im(\psi) = ker(\varphi)$ are P-generated . $Pres(P_R)$ denotes the category of all P-presented modules . Clearly

$$Pres(P_R) \subseteq Gen(P_R) \subseteq \overline{Gen}(P_R) \ .$$

Dually , let $K_A \in Mod\text{-}A$ then $N \in Mod\text{-}A$ is said to be *K-cogenerated* if there exists an exact sequence $0 \rightarrow N \xrightarrow{\psi} P^X$ and N is *K-copresented* if there exists an exact sequence $0 \rightarrow N \xrightarrow{\psi} P^X \xrightarrow{\varphi} P^Y$, where X and Y are sets . Denoted by $Cogen(K_A)$ $(Copres(K_A))$ the subcategory of $Mod\text{-}A$ cogenerated (copresented) by K_A . Clearly $Copres(K_A) \subseteq Cogen(K_A)$.

Recall that a module $N \in Cogen(K_A)$ if and only if $Hom_A(N, K)$ separates the points of N , i.e. for each $x \in N$, $x \neq 0$, there exists a $f \in Hom_A(N, K)$ such that $f(x) \neq 0$. See [AF , 8.11] .

Proposition 1.1.4 *Let $_AP_R$ be any bimodule . Then , with notations above , we have*

1) $Im(T_P) \subseteq Pres(P_R) \subseteq Gen(P_R) \subseteq \overline{Gen}(P_R)$.

2) Let Q_R be a cogenerator of Mod-R , and let $K_A = Hom_R(P, Q)$. Then $Im(H_P) \subseteq Copres(K_A) \subseteq Cogen(K_A)$.

3) σ_N is a monomorphism if and only if $N \in Cogen(K_A)$, where $K_A = Hom_R(P, Q)$ and Q_R is a cogenerator of Mod-R .

4) ρ_M is an epimorphism if and only if $M \in Gen(P_R)$.

5) For every $M \in Mod\text{-}R$, $H_P(M) \cong H_P(t_P(M))$ canonically .

Proof :
1) Let $N \in Mod\text{-}A$, as A_A is a generator of $Mod\text{-}A$, there is an exact sequence

$$A^{(X)} \rightarrow A^{(Y)} \rightarrow N \rightarrow 0$$

for some sets X and Y . Applying the right exact functor $T_P = (- \otimes_A P)$ we get the following exact sequence in $Mod\text{-}R$,

$$(A^{(X)} \otimes_A P) \rightarrow (A^{(Y)} \otimes_A P) \rightarrow T_P(N) \rightarrow 0$$

and since $(A^{(I)} \otimes_A P) \cong P_R^{(I)}$ $[\ ((a_\alpha)_{\alpha\in I}) \otimes p \to (a_\alpha p)_{\alpha \in I}\ ]$ canonically for any set I . We have the commutative diagram with exact rows

$$\begin{array}{ccccccc}
(A^{(X)} \otimes_A P) & \longrightarrow & (A^{(Y)} \otimes_A P) & \longrightarrow & T_P(N) & \longrightarrow & 0 \\
\cong\downarrow & & \downarrow\cong & & \downarrow= & & \\
P_R^{(X)} & \longrightarrow & P_R^{(Y)} & \longrightarrow & T_P(N) & \longrightarrow & 0
\end{array}$$

, here the upper row is exact and so is the lower row . Hence $Im(T_P) \subseteq Pres(P_R)$. Other inclusions are clear .

2) For each $M \in Mod\text{-}R$ there is an exact sequence in $Mod\text{-}R$:

$$0 \to M \to Q^X \to Q^Y$$

where X and Y are sets . Applying the left exact functor $H_P = Hom_R(P, -)$, we have the exact sequence

$$0 \to H_P(M) \to Hom_R(P, Q^X) \to Hom_R(P, Q^Y) \ .$$

And since $Hom_R(P, Q^I) \cong Hom_R(P, Q)^I = K_A^I$ canonically for every set I , we have the commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & H_P(M) & \longrightarrow & Hom_R(P, Q^X) & \longrightarrow & Hom_R(P, Q^Y) \\
 & & =\downarrow & & \downarrow\cong & & \downarrow\cong \\
0 & \longrightarrow & H_P(M) & \longrightarrow & Hom_R(P, Q)^X & \longrightarrow & Hom_R(P, Q)^Y
\end{array}$$

, here the upper row is exact and so is the lower row . Therefore

$$Im(H_P) \subseteq Copres(K_A) \subseteq Cogen(K_A) \ .$$

3) Note that $N \in Cogen(K_A)$ if and only if for each $x \in N$, $x \neq 0$, there exists a $f \in Hom_A(N, K)$ such that $f(x) \neq 0$.

Hence it suffices to prove that σ_N is monic if and only if for each $x \in N$, $x \neq 0$, there exists a $f \in Hom_A(N, K)$ such that $f(x) \neq 0$.

($\Leftarrow$) Note that for $\Phi =$ adjoint isomorphism , see 1.1.2 , we have

$$Hom_A(N, K) = Hom_A(N, Hom_R(P, Q)) \underset{\cong}{\xrightarrow{\Phi}} Hom_R(N \otimes_A P, Q)$$

. Let $0 \neq n \in N$ and then by hypothesis , there is a $f \in Hom_A(N, K)$ such that $f(n) \neq 0$. So there is a $p \in P$ such that $f(n)(p) \neq 0$. But $f = \Phi^{-1}(\beta)$ for some $\beta \in Hom_R(N \otimes_A P, Q)$. We have $0 \neq f(n)(p) = \Phi^{-1}(\beta)(n)(p) = \beta(n \otimes p)$ and hence $n \otimes p \neq 0$ i.e. $\sigma_N(n)(p) = n \otimes p \neq 0$. We have $\sigma_N(n) \neq 0$ and so σ_N is monic .

($\Rightarrow$) Let $0 \neq n \in N$ and then by assumption , $\sigma_N(n) \neq 0$ and so we have $\sigma_N(n)(p) \neq 0$ for some $p \in P$ i.e. $n \otimes p \neq 0$. Note that Q_R cogenerates $N \otimes_A P$, so there exists a $g \in Hom_R(N \otimes_A P, Q)$ s.t. $g(n \otimes p) \neq 0$. But again

$$Hom_A(N, K) = Hom_A(N, Hom_R(P, Q)) \xrightarrow[\cong]{\Phi} Hom_R(N \otimes_A P, Q)$$

and so $g = \Phi(\xi)$ for some $\xi \in Hom_A(N, Hom_R(P, Q)) = Hom_A(N, K)$. Then $g(n \otimes p) = \Phi(\xi)(n \otimes p) = \xi(n)(p) \neq 0$. So $\xi(n) \neq 0$ and therefore K cogenerates N .

4) Note that $Im(\rho_M) = t_P(M) = \{ \sum f(P) \mid f \in Hom_R(P, M)\}$ and $t_P(M) = M$ if and only if $M \in Gen(P_R)$. So ρ is epic if and only if $M \in Gen(P_R)$.

5) Clear . $\Box$

Let $G : \mathcal{C}_A \rightleftarrows \mathcal{G}_R : F$ be an equivalence , where $\mathcal{C}_A$ and $\mathcal{G}_R$ are subcategories of $Mod\text{-}A$ and $Mod\text{-}R$ respectively . If this equivalence is representable i.e. there exists a bimodule ${}_AP_R$ such that

$$G \cong (- \otimes_A P)|_{\mathcal{C}_A} = T_P|_{\mathcal{C}_A}$$

$$F \cong Hom_R(P_R, -)|_{\mathcal{G}_R} = H_P|_{\mathcal{G}_R}$$

, then T_P and H_P define an equivalence

$$T_P : \mathcal{C}_A \rightleftarrows \mathcal{G}_R : H_P$$

and in this case , there are some natural isomorphisms

$$T_P H_P \xrightarrow{\eta} Id_{\mathcal{G}_R}$$

$$Id_{\mathcal{C}_A} \xrightarrow{\zeta} H_P T_P$$

. Now we have the following important

Proposition 1.1.5

(1) There is a natural isomorphism $T_PH_P \xrightarrow{\eta} Id_{\mathcal{G}_R}$ *if and only if* $\rho : T_PH_P \to Id_{\mathcal{G}_R}$ *is a natural isomorphism .*

(2) There is a natural isomorphism $Id_{\mathcal{C}_A} \xrightarrow{\zeta} H_PT_P$ *if and only if* $\sigma : Id_{\mathcal{C}_A} \to H_PT_P$ *is a natural isomorphism .*

Consequently we have that $T_P : \mathcal{C}_A \rightleftarrows \mathcal{G}_R : H_P$ *defines an equivalence of categories if and only if the associated natural morphisms* σ *and* ρ *are isomorphisms .*

Because of this , the natural morphisms σ and ρ are fundamental tools in the categorical theory of modules . To study the equivalence $T_P : \mathcal{C}_A \rightleftarrows \mathcal{G}_R : H_P$, we need only to calculate σ and ρ . See also [S2 , lemma 1.2] and [MO , proposition 2.4] .

Proof :
Now $F \cong T_P$, $G \cong H_P$ and $FG \cong Id_{\mathcal{G}_R}$, $Id_{\mathcal{C}_A} \cong GF$. So we have

$$\eta : T_PH_P \cong Id_{\mathcal{G}_R}$$

$$\zeta : Id_{\mathcal{C}_A} \cong H_PT_P$$

for some natural isomorphisms η and ζ .

(1) Suppose that η is a natural isomorphism and then we need to prove that ρ is also a natural isomorphism .

Let $M_R \in \mathcal{G}_R = ImT_P \subseteq Gen(P_R)$. By proposition 1.1.4(4) , ρ_M is an epimorphism .

By assumption , we have an isomorphism

$$\eta_M : T_PH_P(M) \to M$$

and for $\eta_M \in Hom_R(T_PH_P(M), M) \underset{\cong}{\overset{\Phi}{\leftarrow}} Hom_A(H_P(M), H_P(M))$, here $\Phi =$ adjoint isomorphism of the adjoint pair (H_P, T_P) and see 1.1.2 for the definition of Φ , there is a $\beta \in Hom_A(H_P(M), H_P(M))$ such that $\Phi(\beta) = \eta_M$ and

$$(i) \quad \beta(f)(p) = \Phi(\beta)(f \otimes p) = \eta_M(f \otimes p)$$

for every $f \in Hom_R(P, M)$ and $p \in P$. But

$$Hom_A(H_P(M), H_P(M)) \overset{H_P}{\cong} Hom_R(M, M)$$

and so there is a $h \in Hom_R(M, M)$ s.t. $Hom_R(P, h) = H_P(h) = \beta$.

Then for every $f \in Hom_R(P, M) = H_P(M)$, $\beta(f) = h \circ f$ and hence for every $p \in P$ we have

$$(ii) \quad \beta(f)(p) = h(f(p)) = (h \circ \rho_M)(f \otimes p)$$

. It follows from (i) and (ii) that

$$\eta_M(f \otimes p) = (h \circ \rho_M)(f \otimes p)$$

for every $f \in H_P(M) = Hom_R(P, M)$ and $p \in P$. But then $\eta_M = h \circ \rho_M$ and η_M is an isomorphism , so ρ_M is monic and hence an isomorphism .

(2) Now we prove that if ζ is a natural isomorphism then σ is a natural isomorphism . To see this , let

$$N \in \mathcal{C}_A = Im(H) \subseteq Copres(K_A) \subseteq Cogen(K_A)$$

and then σ_N is monic by proposition 1.1.4(3) .

By assumption , let $\zeta_N : N \to Hom_R(P_R, N \otimes_A P) = H_P T_P(N)$ be an isomorphism . So

$$\zeta_N \in Hom_A(N, Hom_R(P, N \otimes_A P)) \underset{\cong}{\xrightarrow{\Phi}} Hom_R(N \otimes_A P, N \otimes_A P)$$

and then $\Phi(\zeta_N)(n \otimes p) = [\zeta_N(n)](p)$ for every $n \in N$ and $p \in P$. Note that

$$Hom_R(N \otimes_A P, N \otimes_A P) \overset{T_P}{\cong} Hom_A(N, N)$$

, so there is a $h \in Hom_A(N, N)$ such that $T_P(h) = h \otimes Id_P = \Phi(\zeta_N)$ and for every $n \in N$ and $p \in P$, we have

$$[(\sigma_N \circ h)(n)](p) = h(n) \otimes p = T_P(h)(n \otimes p) = \Phi(\zeta_N)(n \otimes p) = [\zeta_N(n)](p)$$

. Hence $\sigma_N \circ h = \zeta_N$ which is an isomorphism and then σ_N is epic . Therefore σ_N is an isomorphism . $\square$

Let ${}_AP_R$ be a bimodule , Q_R be a cogenerator of $Mod\text{-}R$ and $K_A = Hom_R(P, Q)$. Then we have the following subcategories :

$$Im(T_P) \subseteq Pres(P_R) \subseteq Gen(P_R) \subseteq \overline{Gen}(P_R) \subseteq Mod\text{-}R \ ,$$

$$Im(H_P) \subseteq Copres(K_A) \subseteq Cogen(K_A) \subseteq Mod\text{-}A$$

and we are going to study the equivalences between them in the subsequent chapters .

Chapter 2

Some Classical Results

2.1 Morita Theorem

Let A and R be rings , the characterizations of the equivalence between $Mod\text{-}A$ and $Mod\text{-}R$ was done by Morita [M , 1958] and this is the first work on the theory of equivalence for module categories . One may see [AF] or [Jacobson , Basic Algebra II] for a complete presentation of Morita's results .

Definition 2.1.1 *$P_R \in Mod\text{-}R$ is called a progenerator if it is a finitely generated projective generator of $Mod\text{-}R$.*

Theorem 2.1.2 [Morita , 1958]

(a) Let $Mod\text{-}A \underset{H}{\overset{T}{\rightleftarrows}} Mod\text{-}R$ be an equivalence of categories . Then there exists a faithfully balanced bimodule ${}_AP_R$ (${}_AP_R \overset{def}{=} T({}_AA_A)$) which is a progenerator on both sides i.e. ${}_AP$ and P_R are progenerators and

$$\begin{array}{rcl} T & \cong & (- \otimes_A P) = T_P \\ H & \cong & Hom_R(P, -) = H_P \end{array}$$

By proposition 1.1.5 , we know that T_P and H_P define an equivalence between $Mod\text{-}A$ and $Mod\text{-}R$ and their associated natural morphisms σ and ρ are isomorphisms .

This is a representation theorem and the representing modules are progenerators P i.e. every abstract equivalence between $Mod\text{-}A$ and $Mod\text{-}R$ is represented by (T_P, H_P) . Hence to study equivalences between $Mod\text{-}A$ and $Mod\text{-}R$, it is required only to study the equivalences induced by (T_P, H_P) and this means that we just need to consider their associated natural morphisms σ and ρ .

(b) Let ${}_AP_R$ be a bimodule , $T_P = (- \otimes_A P)$ and $H_P = Hom_R(P, -)$.

Then P_R is a progenerator and $A \cong End(P_R)$ canonically if and only if

$$Mod\text{-}A \underset{H_P}{\overset{T_P}{\rightleftarrows}} Mod\text{-}R$$

defines an equivalence via the canonical natural isomorphisms

$$\sigma : Id_{Mod\text{-}A} \to H_P T_P \quad \textit{and} \quad \rho : T_P H_P \to Id_{Mod\text{-}R}$$

Proof :
See [AF , Chapter 6] or [Jacobson , Basic Algebra II , 3.15] for a proof by the notion of Morita context . □

2.2 Fuller Theorem

In fact there are many ways to generalize Morita's theorem in the literature . Our direction is to investigate the equivalences between *subcategories* of modules . The first attempt to generalize Morita's theorem in this direction is due to K. R. Fuller [F , 1974] . He considers the equivalence between the category $Mod\text{-}A$ and some " closed " subcategory of $Mod\text{-}R$.

Definition 2.2.1

(a) $P \in Mod\text{-}R$ is a selfgenerator if it generates each of its submodules .

(b) $P \in Mod\text{-}R$ is quasi-projective if P_R is P_R-projective . That is for any epimorphism $\varphi : P \to M$ in Mod-R and for each $f \in Hom_R(P, M)$ there exists a morphism $g \in Hom_R(P, P)$ (lifting of f) such that $\varphi \circ g = f$.

$$\begin{array}{ccccc} P & \xrightarrow{=} & P & & \\ {\scriptstyle \exists g}\downarrow & & \downarrow{\scriptstyle f} & & \\ P & \xrightarrow{\varphi} & M & \longrightarrow & 0 \end{array}$$

Parallel to the Morita's theorem , we divide Fuller's result into two parts and the first part is a representation theorem .

Theorem 2.2.2 [Fuller , 1974]

(a) (representation theorem) Let $\mathcal{G}_R$ be a closed (complete additive) subcategory of Mod-R (i.e. a full subcategory closed under taking direct sums , epimorphic images and submodules) , and let

$$Mod\text{-}A \underset{H}{\overset{T}{\rightleftarrows}} \mathcal{G}_R$$

be an equivalence .

Then there exists a bimodule ${}_AP_R$ such that $A \cong End(P_R)$ canonically , P_R is a quasi-progenerator and $\mathcal{G}_R = Gen(P_R)(= \overline{Gen}(P_R))$. Moreover

$$\begin{aligned} T &\cong (- \otimes_A P) = T_P|_{Mod\text{-}A} \\ H &\cong Hom_R(P, -) = H_P|_{\mathcal{G}_R} \end{aligned}$$

So , again , to study abstract equivalences between $Mod\text{-}A$ and $\mathcal{G}_R$ we need only to consider the equivalence induced by (T_P, H_P) .

(b) Let ${}_AP_R$ be a bimodule , $T_P = (- \otimes_A P)$ and $H_P = Hom_R(P, -)$. Then P_R is a quasi-progenerator and $A \cong End(P_R)$ canonically if and only if the pair of functors (T_P, H_P) defines an equivalence between Mod-A and $Gen(P_R)$ via the associated natural isomorphisms

$$\sigma : Id_{Mod-A} \to H_P T_P$$

$$\rho : T_P H_P \to Id_{Gen(P_R)}$$

And in this case , $Gen(P_R) = \overline{Gen}(P_R)$ so that $Gen(P_R)$ is a closed subcategory of Mod-R .

We prove this theorem along with many other things .

Proof :
Part (a) See also [F , theorem 1.1] .

Now we have that
(1) H and T are full and faithful functors .
(2) (T, H) and (H, T) are adjoint pairs with respect to the two categories $Mod\text{-}A$ and $\mathcal{G}_R$.

In fact (1) and (2) are ture for all functors that realize a category equivalence [AF , 21.2 and 21.3] .

(3) Define $T(A)_R = P_R$ and then P has an canonical bimodule structure ${}_AP_R$ defined by the composition of ring homomorphisms :

$$A \xrightarrow{cano} End(A_A) \xrightarrow{T} End(T(A)_R)$$

i.e. we use this ring homomorphism to define the left A-scalar multiplication of ${}_AP_R$ [AF , 20.3] . However , since A has an faithfully balanced bimodule structure ${}_AA_A$ and now T is full and faithful , we have $A \cong End(P_R)$ canonically .

(4) we have the A-isomorphisms :

$$\begin{aligned} H(M) &\cong Hom_A(A_A, H(M)) \quad \text{(by [AF , 20.1])} \\ &\cong Hom_R(T(A), M) \quad \text{(as } (T, H) \text{ is an adjoint pair)} \end{aligned}$$

that are natural in $M \in Mod\text{-}R$. Therefore $H \cong Hom_R({}_AP_R, -)$.

Consider the functor $T_P = (-\otimes_A P) : Mod\text{-}A \to Mod\text{-}R$. By assumption , $\mathcal{G}_R$ is closed under direct sums and epimorphic images and $T(A) = P \in \mathcal{G}_R$, so $Im(T_P) = Im(-\otimes_A P) \subseteq Gen(P_R) \subseteq \mathcal{G}_R$.

Now both T and $T_P = (-\otimes_A P)$ are left adjoints of $H \cong Hom_R(P, -) = H_P$, so we have that $T \cong T_P$. See [W , 45.4] or [St , chapter IV , proposition 9.1] .

(5) $P_R = T(A) \in \mathcal{G}_R$, so $Gen(P_R) \subseteq \mathcal{G}_R$. Now let $M_R \in \mathcal{G}_R$, then there exists an epimorphism $A^{(X)} \to H(M)_A \to 0$. Applying the right exact functor T , we get the epimorphism

$$T(A^{(X)}) \to TH(M) \to 0$$

and since $P_R^{(X)} \cong A^{(X)} \otimes_A P = T_P(A^{(X)}) \cong T(A^{(X)})$ and $TH(M) \cong M$, there is an epimorphism $P_R^{(X)} \to M \to 0$ and so $\mathcal{G}_R \subseteq Gen(P_R)$.

Finally we obtain that $\mathcal{G}_R = Gen(P_R) = \overline{Gen}(P_R)$.

(6) We need to show that P_R is a quasi-progenerator . We will show this via Azumaya's approach . See theorem 2.2.5 below .

Part (b)
See also theorem 2.2.5 . □

If there is a bimodule ${}_AP_R$, with $A \cong End(P_R)$ canonically , such that $\mathcal{G}_R = Gen(P_R)(= \overline{Gen}(P_R))$ and

$$\begin{aligned} T &\cong (- \otimes_A P) = T_P|_{Mod\text{-}A} \\ H &\cong Hom_R(P, -) = H_P|_{\mathcal{G}_R} \end{aligned}$$

Then , since (T, H) defines an equivalence between $Mod\text{-}A$ and $Gen(P_R)$ and by proposition 1.1.5 , (T_P, H_P) defines an equivalence between $Mod\text{-}A$ and $Gen(P_R)$ and this means that their associated natural morphisms σ and ρ are isomorphisms .

We now present a refinement of Fuller's theorem [F , theorem 2.6] due to Azumaya [A , 1978] . And **part (a)(6)** and **part (b)** of **Theorem 2.2.2** are contained in Azumaya's theorem . Actually Azumaya found out some characterizations for a bimodule ${}_AP_R$ which induces an equivalence between $Mod\text{-}A$ and $Gen(P_R)$.

Let ${}_AP_R$ be a bimodule . Denote $T_P = (- \otimes_A P)$, $H_P = Hom_R(P, -)$ and the associated natural morphisms $\sigma : Id_{Mod\text{-}A} \to H_PT_P$ and $\rho : T_PH_P \to Id_{Mod\text{-}R}$ as usual . First we give some remarks and definitions .

2.2.3 For a R-module M , $Hom_R(P, M)$ is canonically isomorphic to $Hom_R(P, t_P(M))$ and since $\rho(t_P(M))$ is always epic and $\rho(M)$ is monic if and only if $\rho(t_P(M))$ is monic , we have $\rho(t_P(M))$ is an isomorphism if and only if $\rho(M)$ is monic .

Note also that $M \in Gen(P_R)$ if and only if $M = t_P(M)$. So ρ_M is an isomorphism for every $M \in Gen(P_R)$ if and only if ρ_M is an monomorphism for every $M \in Mod\text{-}R$.

Definition 2.2.4

(1) *We call* ${}_AP$ *a weak generator if* $N \otimes_A P = 0$ *implies* $N = 0$ *for any* $N \in Mod\text{-}A$.

(2) P_R *is called* $\sum$*-quasi-projective if* $Hom_R(P, -)$ *preserves the exactness of* $P_R^{(I)} \to M_R \to 0$ *for every set* I *and* $M \in Mod\text{-}R$.

(3) P_R *is called semi-*$\sum$*-quasi-projective if* $Hom_R(P, -)$ *preserves the exactness of*

$$P_R^{(J)} \to P_R^{(I)} \to M_R \to 0$$

for sets J , I *and* $M \in Mod\text{-}R$.

If an epimorphism $P^{(I)} \xrightarrow{h} M \longrightarrow 0$ is given , here I is a set , and let $\lambda_i : P_R \to P_R^{(I)}$ be the canonical injection , $i \in I$. Then for $h \circ \lambda_i : P_R \to M_R$, we denote $h \circ \lambda_i = h_i \in Hom_R(P, M)_A$ and write $h = (h_i)_{i \in I}$ where

$$h((p_i)) \overset{def}{=} \sum_{i \in I} h_i(p_i) \quad , \quad \text{for } (p_i) \in P_R^{(I)} \quad .$$

Theorem 2.2.5 [Azumaya , 1978] *The following statements are equivalent for a bimodule* ${}_AP_R$ *:*

(1) $\sigma(N)$ *is an isomorphism for every* $N \in Mod\text{-}A$ *and* $Gen(P_R)$ *is closed under taking submodules , i.e.* $Gen(P_R) = \overline{Gen}(P_R)$.

(2) $\sigma(N)$ *is an isomorphism for every* $N \in Mod\text{-}A$ *and* $\rho(M)$ *is a monomorphism for every* $M \in Mod\text{-}R$ *i.e.* T_P *and* H_P *define an equivalence between Mod-A and* $Gen(P_R)$.

(3) ${}_AP$ *is a weak generator and* $\rho(M)$ *is a monomorphism for every* $M \in Mod\text{-}R$.

(4) P_R *is finitely generated , quasi-projective and* $A \cong End(P_R)$ *canonically and* $Gen(P_R) = \overline{Gen}(P_R)$.

(5) P_R *is finitely generated , quasi-projective and generates each of its submodules (i.e.* P_R *is a quasi-progenerator) and* $A \cong End(P_R)$ *canonically .*

Azumaya proved this theorem by a series of lemmas and propositions .

Lemma 2.2.6 [A , lemma 1 and proposition 5] *Let $_AP_R$ be a bimodule and $P_R^{(I)} \xrightarrow{h} M_R \to 0$ be exact and $\lambda_i : P_R \to P_R^{(I)}$ be the canonical injection , $i \in I$. For $h \circ \lambda_i : P_R \to M_R$, we denote $h \circ \lambda_i = h_i \in Hom_R(P,M)_A$ and write $h = (h_i)_{i\in I}$ where*

$$h((p_i)) \stackrel{def}{=} \sum_{i\in I} h_i(p_i) \quad , \quad for \quad (p_i) \in P_R^{(I)}$$

. *Now assume that $\rho(M)$ is a monomorphism then*

(a) *$Hom_R(P,M) \otimes_A P = (\sum_{i\in I} h_i A) \otimes_A P$ and*

(b) *If $_AP$ is a weak generator , then $Hom_R(P,M)_A = \sum_{i\in I}(h_iA)$, that is , for each $f \in Hom_R(P,M)$ there exists $(a_i)_{i\in I} \in A^{(I)}$ such that $f = \sum_{i\in I}(h_ia_i)$ and $f(p) = h((a_ip)_{i\in I})$ for $p \in P$.*

Proof :
Recall that $Hom_R(_AP_R, M_R)_A$ is a right A-modules canonically [AF , 4.4] .
" $\supseteq$ " Clear .
" $\subseteq$ " Let $t \in Hom_R(P,M) \otimes_A P$ and $\rho(M)(t) = x \in M$.
Since h is epic , there exists $(p_i) \in P_R^{(I)}$ such that

$$x = h(p_i) = \sum_{i\in I} h_i(p_i) \quad .$$

But $(h_i \otimes p_i) \in Hom_R(P,M) \otimes_A P$ for all $i \in I$, so $\sum_{i\in I}(h_i \otimes p_i) \in Hom_R(P,M) \otimes_A P$ and

$$\rho(M)(\sum(h_i \otimes p_i)) = \sum_{i\in I} h_i(p_i) = x$$

. By assumption , $\rho(M)$ is monic , so $t = \sum(h_i \otimes p_i) \in Hom_R(P,M) \otimes_A P$.

(b) Now let $L = (Hom_R(P,M)/(\sum h_iA)$ and consider the exact sequence

$$0 \to \sum_{i\in I}(h_iA) \xrightarrow{j} Hom_R(P,M) \xrightarrow{q} L \to 0$$

where $q =$ *natural epimorphism* and $j =$ *inclusion map* . Applying $(- \otimes_A P)$, we get the exact sequence

$$\sum_{i\in I}(h_iA) \otimes_A P \xrightarrow{j\otimes Id_P} Hom_R(P,M) \otimes_A P \xrightarrow{q\otimes Id_P} L \otimes_A P \longrightarrow 0$$

but , by part (a) , $j \otimes Id_P$ is an epimorphism and so $L \otimes_A P = 0$. Then , as ${}_AP$ is a weak generator by hypothesis , $L = 0$ and hence $Hom_R(P, M)_A = \sum_{i\in I}(h_iA)$. □

Proposition 2.2.7 [A , proposition 2] *Let ${}_AP_R$ be a bimodule and ${}_AP$ be flat i.e. $T_P = (- \otimes_A P)$ is exact , and let $\rho(M)$ be monic for all $M \in Mod\text{-}R$. Then $Gen(P_R)$ is closed under submodules i.e. $Gen(P_R) = \overline{Gen}(P_R)$.*

Proof :
Let $L \in \overline{Gen}(P_R)$, so for some set I and N_R , there is an epi $P_R^{(I)} \xrightarrow{\varphi} N_R \to 0$ and $L \le N$. Now $\varphi^{-1}(L) \le P_R^{(I)}$ and $\varphi : \varphi^{-1}(L) \to L \to 0$ is epic and so , if $\varphi^{-1}(L) \in Gen(P_R)$ which is closed under epimorphic image , we have $L \in Gen(P_R)$.

Therefore it suffices to show that for every $K \le P_R^{(I)}$, here I is a set , we have $K \in Gen(P_R)$.

Denote $M = P^{(I)}/K$ and we get the exact sequence

$$0 \to K \xrightarrow{inc} P_R^{(I)} \xrightarrow{h} M \to 0$$

where $h = (h_\alpha)_{\alpha\in I}$, $h_\alpha \in Hom_R(P, M)_A$, is the natural epimorphism and $inc = inclusion\ map$.

For the set I , consider the epimorphism $A^{(I)} \xrightarrow{\phi} (\sum_{\alpha\in I} h_\alpha A) \to 0$ defined by $\phi((a_\alpha)_{\alpha\in I}) \stackrel{def}{=} \sum h_\alpha a_\alpha$ where $(a_\alpha)_{\alpha\in I} \in A^{(I)}$.

Since A_A is a generator of $Mod\text{-}A$ and for $ker(\phi) \le A^{(I)}$, we have the exact sequence

$$A^{(Y)} \xrightarrow{\psi} A^{(I)} \xrightarrow{\phi} (\sum_{\alpha\in I} h_\alpha A) \to 0$$

where Y is an set and $A^{(Y)} \xrightarrow{\psi} Ker(\phi)$ is epic . Applying the right exact functor $(- \otimes_A P) = T_P$, we get the exact sequence

$$A^{(Y)} \otimes_A P \xrightarrow{\psi\otimes Id_P} A^{(I)} \otimes_A P \xrightarrow{\phi\otimes Id_P} (\sum h_\alpha A) \otimes_A P \longrightarrow 0 \ .$$

For $0 \to \sum h_\alpha A \xrightarrow{j} Hom_R(P, M)$, here $j =$ inclusion map , since ${}_AP$ is flat i.e. $(- \otimes_A P)$ is exact and so

$$0 \longrightarrow (\sum h_\alpha A) \otimes_A P \xrightarrow{j\otimes Id_P} Hom_R(P, M) \otimes_A P$$

is a monomorphism . Indeed $j \otimes Id_P$ is an isomorphism by lemma 2.2.6(a) because , by assumption , $\rho(M)$ is monic and

$$h_\alpha \otimes p = (j \otimes Id_P)(h_\alpha \otimes p) \in Hom_R(P, M) \otimes_A P$$

for every $\alpha \in I$ and $p \in P$.

Note that $M \in Gen(P_R)$, so $\rho(M) : Hom_R(P, M) \otimes_A P \to M$ is an isomorphism and then $\beta \overset{def}{=} \rho(M) \circ (j \otimes Id_P) : (\sum h_\alpha A) \otimes_A P \to M$ is an isomorphism .

And now we have the commutative diagram ,

$$\begin{array}{ccccccc} A^{(Y)} \otimes_A P & \xrightarrow{\psi \otimes Id_P} & A^{(I)} \otimes_A P & \xrightarrow{\phi \otimes Id_P} & (\sum h_\alpha A) \otimes_A P & \longrightarrow & 0 \\ \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle q_2} & & \downarrow{\scriptstyle \beta} & & \\ P_R^{(Y)} & \xrightarrow{g} & P_R^{(I)} & \xrightarrow{h} & M & \longrightarrow & 0 \end{array}$$

where q_1 and q_2 are the canonical isomorphisms $A^{(Y)} \otimes_A P \cong P_R^{(Y)}$ and $A^{(I)} \otimes_A P \cong P_R^{(I)}$ respectively and here $g \overset{def}{=} q_2 \circ (\psi \otimes Id_P) \circ q_1^{-1}$ so that the left hand side square is commutative . The right hand side square is commutative because

$$\begin{aligned} \beta \circ (\phi \otimes Id_P)((a_\alpha)_{\alpha \in I} \otimes p) &= \beta \circ (\phi((a_\alpha)_{\alpha \in I}) \otimes p) \\ &= \rho(M) \circ (j \otimes Id_P)((\sum h_\alpha a_\alpha) \otimes p) \\ &= \rho(M)((\sum h_\alpha a_\alpha) \otimes p) \\ &= (\sum h_\alpha a_\alpha)(p) \\ &= \sum h_\alpha (a_\alpha p) \\ &= h((a_\alpha p)_{\alpha \in I}) \\ &= h \circ q_2((a_\alpha)_{\alpha \in I} \otimes p) \end{aligned}$$

for every $(a_\alpha)_{\alpha \in I} \in A^{(I)}$ and $p \in P$.

From the diagram , since the upper row is exact , the lower row is also exact and then $ker(h) = K = Im(g) \in Gen(P_R)$. □

Lemma 2.2.8 [A , lemma 3] *Let ${}_AP_R$ be a bimodule such that P_R is quasi-projective and $A \overset{\lambda}{\cong} End(P_R)$ canonically . Then for every finitely generated right ideal $L \leq A_A$, the canonical map*

$$\phi : L_A \to Hom_R(P, LP)_A \quad [x \mapsto [p \mapsto xp]]$$

is an isomorphism .

Proof :
Define $\phi : L_A \to Hom_R(P, LP)_A$ by $\phi(x)(p) = xp$ for every $x \in L$ and $p \in P$. Obviously ϕ is an A-homomorphism and injective .

Now let $L_A = \langle\, x_i \in L \mid i = 1, 2, ..., n \,\rangle_A = \sum_{i=1}^{n} x_i A$. Then

$$LP = \sum_{i=1}^{n} x_i P$$

and we define an epimorphism

$$h : P_R^{(n)} \to (LP)_R$$

by $h((p_1, ..., p_n)) = \sum_i x_i p_i$.

Let $f \in Hom_R(P, PL)_A$, since P_R is P_R-projective , P_R is $P_R^{(n)}$-projective [AF , 16.12] and so there is a $g \in Hom_R(P, P^{(n)})$ such that $hg = f$.

$$\begin{tikzcd}
P \arrow[r, "="] \arrow[d, "\exists\, g"'] & P \arrow[d, "f"] & \\
P_R^n \arrow[r, "h"] & LP \arrow[r] & 0
\end{tikzcd}$$

Write $g = (g_1, ..., g_n)$ for some $g_i \in Hom_R(P, P) \overset{\lambda}{\cong} A$ and then $g(p) = (g_1(p), ..., g_n(p))$ for every $p \in P$. For $i = 1, ..., n$ we write $g_i = \lambda(a_i)$ for some $a_i \in A$ and

$$\begin{aligned}
f(p) &= hg(p) \\
&= h((g_1(p), ..., g_n(p))) \\
&= \sum x_i(a_i p) \\
&= (\sum x_i a_i)(p) \\
&= \phi(\sum x_i a_i)(p)
\end{aligned}$$

for every $p \in P$. But $\sum x_i a_i \in L_A$ and so ϕ is epic . It follows that ϕ is an A-isomorphism . □

Lemma 2.2.9 [A , lemma 4] *Let ${}_AP_R$ be a bimodule and P_R be quasi-projective with $A \cong End(P_R)$ canonically . If $\rho(K)$ is monomorphism for all $K \leq P_R$, then ${}_AP$ is flat .*

Proof :
Recall that ${}_AP$ is flat iff for every finitely generated right ideal $L \leq A_A$ we have that $\mu : L \otimes_A P \to LP$ $[l \otimes p \mapsto lp]$ monic [AF , 19.17(c)] .

Let L be a finitely generated right ideal of A . By lemma 2.2.8 , there is a A-isomorphism $\phi : L_A \to Hom_R(P, LP)_A$ defined by $\phi(x)(p) = xp$ for every $x \in L$ and $p \in P$. Applying $(- \otimes_A P)$, we get an isomorphism

$$\phi \otimes Id_P : L_A \otimes_A P \to Hom_R(P, LP)_A \otimes_A P$$

. By assumption , as $(LP)_R \leq P_R$, $\rho((LP)_R) : Hom_R(P, LP)_A \otimes_A P \to LP$ is a monomorphism . So

$$\rho((LP)_R) \circ (\phi \otimes Id_P) : L_A \otimes_A P \to LP$$

is also a monomorphism . Note that for every $l \in L$ and $p \in P$,

$$\rho((LP)_R) \circ (\phi \otimes Id_P)(l \otimes p) = \rho((LP)_R)(\phi(l) \otimes p) = \phi(l)(p) = lp = \mu(l \otimes p)$$

and hence μ is monic . $\square$

Corollary 2.2.10 [A , corollary 6] *Let ${}_AP_R$ be a bimodule . If ${}_AP$ is a weak generator and $\rho(P_R^{(I)})$ is injective for every set I , then the canonical map*

$$\varphi : A_A^{(I)} \to Hom_R(P, P^{(I)})_A \qquad [(a_i) \mapsto [p \mapsto (a_ip)]]$$

is an isomorphism and in particular $A \cong End(P_R)$ canonically .

Proof :
Note that if ${}_AP$ is a weak generator , then ${}_AP$ is faithful as a A-module because for $a \in A$ and $aP = 0$, then $aA \otimes_A P = (1 \otimes_A (aP)) = 0$ and so $aA = 0$. Therefore $a = 0$.

For any set I , define $\varphi : A_A^{(I)} \to Hom_R(P, P^{(I)})_A$ by $\varphi((a_i)_{i \in I})(p) = (a_ip)_{i \in I}$ for every $(a_i)_{i \in I} \in A^{(I)}$ and $p \in P$.

φ is injective because ${}_AP$ is faithful . For if $a \in A$ and $aP = 0$ then $a = 0$

.

Now consider $P_R^{(I)} \xrightarrow{Id} P_R^{(I)}$ where Id = *identity map* . By assumption , $\rho(P_R^{(I)})$ is monic and by lemma 2.2.6(b) , for each $f \in Hom_R(P, P_R^{(I)})$, there exists $(a_i)_{i \in I} \in A^{(I)}$ such that $f(p) = Id((a_ip)_{i \in I})$ for every $p \in P$. Therefore

$$f(p) = (a_ip)_{i \in I} = \varphi((a_i)_{i \in I})(p) \quad for \quad p \in P$$

and so φ is epic . $\square$

Corollary 2.2.11 [A , corollary 7] *Let ${}_AP_R$ be a bimodule , ${}_AP$ be a weak generator and $\rho(M)$ be a monomorphism for every $M \in Mod\text{-}R$. Then P_R is $\sum$-quasi-projective i.e. P_R is $P_R^{(I)}$-projective for every set I .*

Proof :
Let $\lambda : A \longrightarrow End(P_R)$ be the canonical ring homomorphism (A-scalar left multiplication) . For any set I , let $h : P_R^{(I)} \longrightarrow M_R$ be an epimorphism and $f : P_R \longrightarrow M_R$, then by lemma 2.2.6(b) , there exists $(a_i)_{i\in I} \in A^{(I)}$ such that $f(p) = h((a_ip)_{i\in I})$ for $p \in P$. Note that $a_i \neq 0$ for only finitely many $i \in I$ and so

$$g : P \longrightarrow P^{(I)} \quad [p \mapsto (a_ip)_{i\in I}]$$

is well defined and we have $hg = f$.

$$\begin{tikzcd} P \arrow[d, "g"'] & P \arrow[l, "="'] \arrow[d, "f"] & \\ P^{(I)} \arrow[r, "h"] & M \arrow[r] & 0 \end{tikzcd}$$

Finally we have checked that P_R is $\sum$-quasi-projective . $\square$

Proposition 2.2.12 [A , proposition 8] *Let ${}_AP_R$ be a bimodule . Then the following conditions are equivalent :*

(a) P_R is finitely generated , quasi-projective and $A \cong End(P_R)$ canonically .

(b) P_R is $\sum$-quasi-projective and for every set I ,

$$A_A^{(I)} \cong Hom_R(P, P^{(I)})_A \quad [(a_i) \mapsto [p \mapsto (a_ip)]]$$

canonically .

Proof :
See also [S1 , theorem 3.1] .
(b) $\Rightarrow$ (a) Let $\{K_i \mid i \in I\}$ be the family of all cyclic submodules of P_R and consider the epimorphism

$$h : \bigoplus_{i\in I} K_i \to P_R$$

defined by $h((k_i)_{i\in I}) = \sum_{i\in I} k_i$ for every $(k_i)_{i\in I} \in \bigoplus_{i\in I} K_i$. By assumption , P_R is $P_R^{(I)}$-projective . Since $\bigoplus_{i\in I} K_i \leq P_R^{(I)}$, we have that P_R is $(\bigoplus_{i\in I} K_i)$-projective [AF , 16.12(1)] .

So for $Id_P = identity\ map : P_R \to P_R$, there exists a $g \in Hom_R(P, \bigoplus_{i\in I} K_i)$ such that $hg = Id_P$.

$$\begin{tikzcd} P \arrow[r, "="] \arrow[d, "\exists\ g"'] & P \arrow[d, "Id_P"] & \\ \bigoplus_{i\in I} K_i \arrow[r, "h"] & P \arrow[r] & 0 \end{tikzcd}$$

We can regard g as a homomorphism $P_R \to P_R^{(I)}$ (i.e. we compose g with the inclusion map $\bigoplus_{i\in I} K_i \hookrightarrow P_R^{(I)}$.)

Given that $A_A^{(I)} \cong Hom_R(P, P^{(I)})_A$ canonocally , so there exists $(a_i)_{i\in I} \in A^{(I)}$ and for every $p \in P$, $g(p) = (a_i p)_{i\in I} \in \bigoplus_{i\in I} K_i$. Then $a_i P \leq K_i$ for all $i \in I$ and since $(a_i)_{i\in I} \in A^{(I)}$, there is a finite set $F \subseteq I$ such that $a_i = 0$ for every $i \notin F$. Now for each $p \in P$,

$$p = h(g(p)) = h((a_i p)_{i\in I}) = \sum_{i\in I}(a_i p) = \sum_{i\in F}(a_i p) \in \sum_{i\in F} K_i$$

and thus $P_R = \sum_{i\in F} K_i$ which is finitely generated . Also a $\sum$-quasi-projective module is clearly quasi-projective .

(a) $\Rightarrow$ (b) Since P_R is quasi-projective and *finitely generated* , P_R is $\sum$-quasi-projective [AF , 16.12] .

As P_R is finitely generated and $A \cong End(P_R)$ canonically , we have

$$A^{(I)} \cong (End(P_R))^{(I)} = (Hom_R(P, P))^{(I)} \cong Hom_R(P, P^{(I)})$$

canonically . □

Proposition 2.2.13 [A , proposition 9] *Let ${}_AP_R$ be a bimodule , σ and ρ be the associated natural morphisms of functors $T_P = (- \otimes_A P)$ and $H_P = Hom_R(P, -)$. If $\sigma(N)$ is an epimorphism for every $N \in Mod\text{-}A$ and $Gen(P_R)$ is closed under submodules . Then $\rho(M)$ is a monomorphism for every $M \in Mod\text{-}R$.*

Proof :
For each $N \in Mod\text{-}A$, by lemma 1.1.3(a) , $\rho(T_P(N)) \circ T_P(\sigma(N)) = Id_{T_P(N)}$ and then $T_P(\sigma(N))$ is a monomorphism . By assumption , $\sigma(N)$ is an epimorphism and T_P is a right exact functor , so we have $T_P(\sigma(N))$ is epic whence isomorphic . It follows that $\rho(T_P(N))$ is an isomorphism i.e. $\rho(M_R)$ is an isomorphism whenever $M_R \cong T_P(N_A)$ for $M \in Mod\text{-}R$ and $N \in Mod\text{-}A$.

Let $M \in Gen(P_R)$ we claim that $\rho(M_R)$ is an isomorphism .
First we have an exact sequence

$$0 \to K \to P_R^{(I)} \xrightarrow{g} M \to 0$$

for some set I . By assumption $K \in Gen(P_R)$ and so we have an epimorphism $f : P_R^{(J)} \to K$ for some set J and we get the exact sequence

$$P_R^{(J)} \xrightarrow{f} P_R^{(I)} \xrightarrow{g} M \to 0 \ .$$

Applying H_P , we get the exact sequence

$$H_P(P_R^{(J)}) \xrightarrow{H_P(f)} H_P(P_R^{(I)}) \xrightarrow{\pi} Coker(H_P(f)) \to 0$$

and then applying T_P , we have the commutative diagram with exact rows

$$\begin{tikzcd}
T_PH_P(P_R^{(J)}) \arrow[r, "TH(f)"] \arrow[d, "\rho(P_R^{(J)})"] & T_PH_P(P_R^{(I)}) \arrow[r, "T(\pi)"] \arrow[d, "\rho(P_R^{(I)})"] & T_P(Coker(H_P(f))) \arrow[r] \arrow[d, "\exists\, h"] & 0 \\
P_R^{(J)} \arrow[r, "f"] & P_R^{(I)} \arrow[r, "g"] & M \arrow[r] & 0
\end{tikzcd}$$

where $P_R^{(J)} \cong T_P(A^{(J)})$ and $P_R^{(I)} \cong T_P(A^{(I)})$, so $\rho(P_R^{(J)})$ and $\rho(P_R^{(I)})$ are isomorphisms . Note that

$$\begin{aligned}
Ker(T(\pi)) = Im(TH(f)) &= Im((\rho(P^{(I)}))^{-1} \circ f \circ \rho(P^{(J)})) \\
&= (\rho(P^{(I)}))^{-1}(Im(f)) \\
&= (\rho(P^{(I)}))^{-1}(Ker(g)) \\
&= Ker(g \circ \rho(P^{(I)})
\end{aligned}$$

and then there exists $h : T_P(Coker(H_P(g))) \rightarrow M$ such that the right hand side square is commutative [AF , 3.6(1)] . Consequently h is bijective and $T_P(Coker(H_P(g))) \cong M$. So $\rho(M)$ is an isomorphism for every $M \in Gen(P_R)$. That is we have proved that $\rho(M)$ is a monomorphism for every $M \in Mod\text{-}R$. See also remarks 2.2.3 . $\square$

Lemma 2.2.14 *If P_R is a quasiprojective module and a selfgenerator then* $Gen(P_R) = \overline{Gen}(P_R)$.

Proof :
See also [F , lemma 2.2] and [S1 , lemma 3.2] .
Again it suffices to prove that for every $K \leq P_R^{(I)}$, where I is a set , we have $K \in Gen(P_R)$. If P_R generates each cyclic submodule $zR \leq K$, then P_R generates K also .

Furthermore we can assume K is a cyclic R-module i.e. $K = zR \leq P_R^{(I)}$ and I is a finite set . So now it will be sufficient (by induction) to prove that all submodules of P_R^n , $n \geq 1$, are generated by P_R .

However , this amounts to showing that if P_R is M_i-projective for $i = 1, 2$ and P_R generates all submodules of M_1 and M_2 then P_R generates all submodules of $M_1 \bigoplus M_2$.

Now let $K \leq M_1 \bigoplus M_2$ and define $N = K \cap M_2 \leq M_2$ and we have $N \in Gen(P_R)$ by assumption . Consider the natural epimorphism $\mu : K + M_2 \rightarrow (K + M_2)/(N + M_2)$ and note that $\mu|_K : K \rightarrow (K + M_2)/(N + M_2)$ is epic also and $Ker(\mu|_K) = N$. So we get

$$K/N \cong (K+M_2)/(N+M_2) \leq (M_1+M_2)/(N+M_2) \cong M_1/(M_1 \cap (N+M_2))$$

and then by assumption , $K/N \in Gen(P_R)$. And we have two epimorphisms $f : P^{(J)} \rightarrow N \rightarrow 0$ and $h : P^{(I)} \rightarrow K/N \rightarrow 0$ for some sets J and I .

Now we consider the exact sequence

$$0 \longrightarrow N \xrightarrow{inc} K \xrightarrow{\pi} K/N \longrightarrow 0$$

By assumption , P_R is $(M_1 \bigoplus M_2)$-projective . Since $K \leq M_1 \bigoplus M_2$, P_R is K-projective [AF , 16.12] and we get an epimorphism

$$Hom_R(P, K) \xrightarrow{\pi_*} Hom_R(P, K/N) \rightarrow 0$$

where $\pi_*(\varphi) = \pi \circ \varphi$ for $\varphi \in Hom_R(P, K)$. Let $\lambda_i : P \to P^{(I)}$ be the natural injection for every $i \in I$. Since π_* is epic , for $h \circ \lambda_i : P \to K/N$, there exists $\varphi_i \in Hom_R(P, K)$ such that $\pi\varphi_i = h\lambda_i$, for all $i \in I$.

$$\begin{array}{ccc} P^{(I)} & \xrightarrow{h} & K/N \\ {\scriptstyle \lambda_i}\uparrow & & \uparrow{\scriptstyle \pi} \\ P & \xrightarrow{\exists\ \varphi_i} & K \end{array}$$

Denote $g = \bigoplus_{i\in I} \varphi_i : P^{(I)} \to K$ [AF , 6.6] and we have

$$\pi \circ g = \pi(\bigoplus_{i\in I} \varphi_i) = \bigoplus_{i\in I}(\pi\varphi_i) = \bigoplus_{i\in I}(h\lambda_i) = h \quad .$$

Claim $K = N + Im(g)$: if $x \in K$, since h is epic , $\pi(x) = h(z) = \pi g(z)$ for some $z \in P^{(I)}$. That is $(x - g(z)) \in Ker(\pi) = N$.

Now $N \in Gen(P_R)$ and $Im(g) \in Gen(P_R)$ so $K \in Gen(P_R)$. □

2.2.15 Proof of Theorem 2.2.5 [A , theorem 10 , 1978]

(1) ⇒ (2) By proposition 2.2.13 .

(2) ⇒ (3) If for some $N \in Mod\text{-}A$, $N \otimes_A P = 0$ and then since $\sigma(N)$: $N \cong Hom_R(P, N \otimes P) = 0$ we have $N = 0$. It follows that ${}_AP$ is a weak generator .

(3) ⇒ (4) Suppose that ${}_AP$ is a weak generator and $\rho(M)$ is a monomorphism for each $M \in Mod\text{-}R$.

By corollary 2.2.11 , P_R is $\sum$-quasi-projective and by corollary 2.2.10 , for every set I , $A_A^{(I)} \cong Hom_R(P, P^{(I)})_A$ canonically . Then by proposition 2.2.12 , P_R is finitely generated , quasi-projective and $A \cong End(P_R)$ canonically .

Then by lemma 2.2.9 , ${}_AP$ is flat , further by proposition 2.2.7 , $Gen(P_R) = \overline{Gen}(P_R)$ and now we have the condition (4) .

(4) ⇒ (1) By proposition 2.2.12 , we have P_R is $\sum$-quasi-projective , hence semi- $\sum$-quasi-projective , and $A_A^{(I)} \cong Hom_R(P, P^{(I)})_A$ canonically . Now we

need to prove that $\sigma(N)$ is an isomorphism for each $N \in Mod\text{-}A$.

Now for any set S ,

$$\phi : A_A^{(S)} \cong Hom_R(P, P^{(S)})_A \quad [(a_i)_{i\in S} \longmapsto [p \mapsto (a_i p)_{i\in S}]]$$

and

$$\xi : A^{(S)} \otimes_A P \cong P^{(S)} \quad [\ ((a_i)_{i\in S}) \otimes p \mapsto (a_i p)_{i\in\Lambda}\ ]$$

, so we get an isomorphism

$$\xi_*^{-1} : Hom_R(P, P^{(S)}) \to Hom_R(P, A^{(S)} \otimes_A P)$$

defined by $\xi_*^{-1}(f) = \xi^{-1} \circ f$ for every $f \in Hom_R(P, P^{(S)})$ and note that $\sigma(A^{(S)}) = \xi_*^{-1} \circ \phi$, so $\sigma(A^{(S)})$ is an isomorphism for every set S .

For $N \in Mod\text{-}A$, consider the exact sequence

$$A^{(J)} \to A^{(I)} \to N_A \to 0$$

and applying $(- \otimes_A P)$, we get the commutative diagram with exact rows

$$\begin{tikzcd} A^{(J)} \otimes_A P \arrow[r] \arrow[d, "\cong"] & A^{(I)} \otimes_A P \arrow[r] \arrow[d, "\cong"] & N_A \otimes_A P \arrow[r] \arrow[d, "="] & 0 \\ P_R^{(J)} \arrow[r] & P_R^{(I)} \arrow[r] & N_A \otimes_A P \arrow[r] & 0 \end{tikzcd}$$

. Then applying $H_P = Hom_R(P, -)$ and note that P_R is semi-$\sum$-quasi-projective , we get the commutative diagram with exact rows

$$\begin{tikzcd} H_P(A^{(J)} \otimes_A P) \arrow[r] \arrow[d, "\cong"] & H_P(A^{(I)} \otimes_A P) \arrow[r] \arrow[d, "\cong"] & H_P(N_A \otimes_A P) \arrow[r] \arrow[d, "="] & 0 \\ H_P(P_R^{(J)}) \arrow[r] & H_P(P_R^{(I)}) \arrow[r] & H_P(N_A \otimes_A P) \arrow[r] & 0 \end{tikzcd}$$

Finally we obtain a commutative diagram with exact rows

$$\begin{tikzcd} A^{(J)} \arrow[r] \arrow[d, "\cong"', "\sigma(A^{(J)})"] & A^{(I)} \arrow[r] \arrow[d, "\cong"', "\sigma(A^{(I)})"] & N_A \arrow[r] \arrow[d, "\sigma(N_A)"] & 0 \\ H_P(A^{(J)} \otimes_A P) \arrow[r] & H_P(A^{(I)} \otimes_A P) \arrow[r] & H_P(N_A \otimes_A P) \arrow[r] & 0 \end{tikzcd}$$

where $\sigma(A^{(J)})$ and $\sigma(A^{(I)})$ are isomorphisms and hence $\sigma(N_A)$ is an isomorphism .

(4) $\Rightarrow$ (5) Clear .
(5) $\Rightarrow$ (4) By lemma 2.2.14 . $\square$

We end this section by giving examples of quasi-progenerators which are not progenerators and so Fuller's result is an effective generalization of Morita's theorem .

Example 2.2.16 [MO , Remark 5.8] Let R be a right primitive ring but NOT right artinian [AF , 14.6(2)] . So we get a faithful simple right R-module P_R . Let $A = End(P_R)$, then
(i) P_R is a quasi-progenerator i.e. ${}_AP_R$ induces an equivalence between Mod-A and $Gen(P_R)$ but
(ii) now $Gen(P_R)$ is the category of all semisimple modules of the form $P_R^{(X)}$. Thus , if R is not right artinian then $Gen(P_R) \neq Mod$-R and so P_R is not a progenerator .

2.3 The Equivalence $Mod\text{-}A \sim Im(T_P)$

Let ${}_AP_R$ be a bimodule , $T_P = (- \otimes_A P)$ and $H_P = Hom_R(P_R, -)$ with the associated natural morphisms σ and ρ . One of the main techniques in studying the category equivalence

$$\mathcal{C}_A \underset{H_P}{\overset{T_P}{\rightleftarrows}} \mathcal{G}_R$$

is that first we suppose that T_P and H_P define an equivalence as above and then try to find out some characterizations of the bimodule ${}_AP_R$. In this way , Sato ([S1 , 1978] and [S2 , 1979]) obtained some useful results for the equivalences : Mod-$A \sim Im(T_P)$ and $Im(H_P) \sim Im(T_P)$.

Theorem 2.3.1 [S1 , theorem 2.1] *Let ${}_AP_R$ be a bimodule , $T_P = (- \otimes_A P)$ and $H_P = Hom_R(P_R, -)$ with the associated natural morphisms σ and ρ . Then the following statements are equivalent :*

(1) $\sigma : Id_{Mod\text{-}A} \to H_PT_P$ *is a natural isomorphism .*

(2) T_P *and* H_P *induce an equivalence between* $Mod\text{-}A$ *and* $Im(T_P)$.

(3) T_P *and* H_P *induce an equivalence between* $Mod\text{-}A$ *and* $Pres(P_R)$.

(4) P_R *is semi-*$\sum$*-quasi-projective* , $A \cong End(P_R)$ *canonically and for every set* I , *the canonical map*

$$\phi : Hom_R(P_R, P_R)^{(I)} \longrightarrow Hom_R(P_R, P_R^{(I)}) \quad [(f_i)_{i\in I} \longmapsto [p \mapsto (f_i(p))_{i\in I}]]$$

is an isomorphism .

Proof :
(1) $\Rightarrow$ (2) For every $N \in Mod\text{-}A$, $\sigma(N)$ is an isomorphism and so $T_P(\sigma(N))$ is an isomorphism too . But $\rho(T_P(N)) \circ T_P(\sigma(N)) = Id_{T_P(N)}$, so $\rho(T_P(N))$ is an isomorphism for every $N \in Mod\text{-}A$ i.e. $\rho(M)$ is an isomorphism for all $M \in Im(T_P)$ and hence $\rho : T_P H_P \to Id_{Im(T_P)}$ is a natural isomorphism .

(2) $\Rightarrow$ (3) Since $Im(T_P) \subseteq Pres(P_R)$, it suffices to prove that $Pres(P_R) \subseteq Im(T_P)$. Let $M \in Pres(P_R)$ and then we have an exact sequence

$$P_R^{(J)} \xrightarrow{f} P_R^{(I)} \xrightarrow{g} M_R \longrightarrow 0$$

for some sets J and I .

An argument similar to that in the proof of proposition 2.2.13 gives us a a commutative diagram with exact rows

$$\begin{array}{ccccccc}
T_P H_P(P_R^{(J)}) & \xrightarrow{T_P H_P(f)} & T_P H_P(P_R^{(I)}) & \xrightarrow{T_P(\pi)} & T_P(Coker(H_P(f))) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \rho(P^{(J)})} & & \downarrow{\scriptstyle \rho(P^{(I)})} & & \downarrow{\scriptstyle \exists\, h} & & \\
P_R^{(J)} & \xrightarrow{f} & P_R^{(I)} & \xrightarrow{g} & M_R & \longrightarrow & 0
\end{array}$$

and from this diagram , it follows that h is an isomorphism and hence $M \cong T_P(Coker(H_P(f))) \in Im(T_P)$.

(3) $\Rightarrow$ (4) Note that , by [AF , 20.1(2)] , we have the isomorphism

$$\mu : A \otimes_A P \cong P \quad [a \otimes p \mapsto ap]$$

and since

$$A \overset{\lambda}{\cong} End(A_A) \overset{T_P}{\cong} End(T_P(A)_R) \cong End(P_R)$$

, so we have the isomorphism

$$\Lambda : A \cong End(P_R)$$

defined by $\Lambda(a) \overset{def}{=} \mu \circ (\lambda(a) \otimes Id_P) \circ \mu^{-1}$ for every $a \in A$ and

$$\begin{aligned} \Lambda(a)(p) &= \mu \circ (\lambda(a) \otimes Id_P) \circ \mu^{-1}(p) \\ &= \mu(\lambda(a) \otimes Id_P)(1 \otimes p) \\ &= \mu(a \otimes p) = ap \end{aligned}$$

for every $p \in P$, so Λ just is the canonical ring homomorphism : $A \to End(P_R)$. Hence $A \cong End(P_R)$ canonically .

Also

$$\begin{aligned} Hom_R(P_R, P_R)^{(I)} = (End(P_R))^{(I)} &\overset{(\Lambda^{(I)})^{-1}}{\cong} A^{(I)} \\ &\overset{\sigma(A^{(I)})}{\cong} Hom_R(P_R, A^{(I)} \otimes_A P) \\ &\overset{\varphi_*}{\cong} Hom_R(P_R, P_R^{(I)}) \end{aligned}$$

where $\varphi : A^{(I)} \otimes_A P \to P^{(I)}$ is defined by $\varphi((a_i)_{i\in I} \otimes p) = (a_i p)_{i\in I}$ for every $(a_i)_{i\in I} \in A^{(I)}$ and $p \in P$ and $\varphi_* : Hom_R(P_R, A^{(I)} \otimes_A P) \to Hom_R(P_R, P_R^{(I)})$ is defined by $\varphi_*(f) = \varphi \circ f$ for every $f \in Hom_R(P_R, A^{(I)} \otimes_A P)$.

Now we check that

$$\varphi_* \circ \sigma(A^{(I)} \circ (\Lambda^{(I)})^{-1} = \phi : Hom_R(P_R, P_R)^{(I)} \to Hom_R(P_R, P_R^{(I)})$$

and so the canonical map is an isomorphism .

For every $(h_i)_{i\in I} \in End(P_R)^{(I)}$, there is a unique $(a_i)_{i\in I} \in A^{(I)}$ such that $\Lambda(a_i) = h_i$ for all $i \in I$. And for every $p \in P$,

$$\begin{aligned} [\varphi_* \circ \sigma(A^{(I)} \circ (\Lambda^{(I)})^{-1}((h_i)_{i\in I})](p) &= (\varphi_* \circ \sigma(A^{(I)})((a_i)_{i\in I}))(p) \\ &= \varphi((a_i)_{i\in I} \otimes p) \\ &= (a_i p)_{i\in I} \\ &= (\Lambda(a_i)(p))_{i\in I} \\ &= (h_i(p))_{i\in I} \\ &= \phi((h_i)_{i\in I})(p) \end{aligned}$$

i.e. the composition map is canonical .

Finally we need to prove that P_R is semi-$\sum$-quasi-projective .
Let $P^{(J)} \xrightarrow{f} P^{(I)} \xrightarrow{g} L_R \to 0$ be an exact sequence with $L_R \in Mod\text{-}R$.

As in the proof of proposition 2.2.13 , first apply H_P and we get the exact sequence

$$H_P(P_R^{(J)}) \xrightarrow{H_P(f)} H_P(P_R^{(I)}) \xrightarrow{\pi} Coker(H_P(f)) \to 0$$

. Then applying T_P we have the commutative diagram with exact rows

$$\begin{tikzcd}
T_PH_P(P_R^{(J)}) \arrow[r, "T_PH_P(f)"] \arrow[d, "\rho(P^{(J)})"'] & T_PH_P(P_R^{(I)}) \arrow[r, "T_P(\pi)"] \arrow[d, "\rho(P^{(I)})"] & T_P(Coker(H_P(f))) \arrow[r] \arrow[d, "\exists\, h"] & 0 \\
P_R^{(J)} \arrow[r, "f"] & P_R^{(I)} \arrow[r, "g"] & L_R \arrow[r] & 0
\end{tikzcd}$$

. Clearly $P^{(J)}$ and $P^{(I)} \in Pres(P_R)$, so by assumption , $\rho(P^{(J)})$ and $\rho(P^{(I)})$ are isomorphisms . Note that

$$\begin{aligned}
Ker(T(\pi)) = Im(TH(f)) &= Im((\rho(P^{(I)}))^{-1} \circ f \circ \rho(P^{(J)})) \\
&= (\rho(P^{(I)}))^{-1}(Im(f)) \\
&= (\rho(P^{(I)}))^{-1}(Ker(g)) \\
&= Ker(g \circ \rho(P^{(I)})
\end{aligned}$$

and then there exists a $h : T_P(Coker(H_P(f))) \to L$ such that the right hand side square is commutative and consequently h is an isomorphism .

Now let $N_A = Coker(H_P(f))$. Then $T_P(N) \overset{h}{\cong} L$ and , by condition (3) , σ_N is an isomorphism . Since $H_P(h) : H_PT_P(N) \cong H_P(L)$ and so $H_P(h) \circ \sigma(N) : N \cong H_P(L)$. As H_P is an additive functor , it preserves commutative diagram and isomorphisms and then we have the commutative diagram with exact rows

$$\begin{tikzcd}
H_P(P_R^{(J)}) \arrow[r, "H_P(f)"] \arrow[d, "\sigma_{H(P_R^{(J)})}"'] & H_P(P_R^{(I)}) \arrow[r, "\pi", "epic"'] \arrow[d, "\sigma_{H(P_R^{(I)})}"] & Coker(H_P(f)) = N \arrow[d, "\sigma_N", "\cong"'] \\
H_PT_PH_P(P_R^{(J)}) \arrow[r, "HTH(f)"] \arrow[d, "H(\rho(P_R^{(J)}))"'] & H_PT_PH_P(P_R^{(I)}) \arrow[r, "HT(\pi)"] \arrow[d, "H(\rho(P_R^{(I)}))"] & H_PT_P(N) \arrow[d, "H(h)", "\cong"'] \\
H_P(P_R^{(J)}) \arrow[r, "H(f)"] & H_P(P_R^{(I)}) \arrow[r, "H(g)"] & H_P(L)
\end{tikzcd}$$

. Here , since $\sigma_{H(P_R^{(J)})}$, $\sigma_{H(P_R^{(I)})}$ and σ_N are isomorphisms and π is epic , we get that $H_PT_P(\pi)$ is epic . But then $H_P(g)$ is epic because $H_P(\rho(P_R^{(J)}))$ $H_P(\rho(P_R^{(J)}))$ and $H_P(h)$ are isomorphisms . Hence

$$H_P(P_R^{(J)}) \xrightarrow{H(f)} H_P(P_R^{(I)}) \xrightarrow{H(g)} H_P(L) \longrightarrow 0$$

is exact i.e. H_P preserves the exactness of $P^{(J)} \xrightarrow{f} P^{(I)} \xrightarrow{g} L_R \to 0$.

(4) $\Rightarrow$ (1) By hypothesis (4) , P_R is semi-$\sum$-quasi-projective and $A^{(I)} \cong Hom_R(P, P^{(I)})$ canonically . We need to show that $\sigma : Id_{Mod\text{-}A} \to H_PT_P$ is a natural isomorphism . But it follows from the same proof as that of [theorem 2.2.5 (4) $\Rightarrow$ (1)] . $\square$

2.4 The Equivalence $Im(H_P) \sim Im(T_P)$

Let $_AP_R$ be a bimodule , $T_P = (- \otimes_A P)$ and $H_P = Hom_R(P_R, -)$, clearly $Im(H_P)$ and $Im(T_P)$ are the smallest subcategories between which T_P and H_P may induce an equivalence i.e. if we have an equivalence

$$T_P : \mathcal{C}_A \rightleftarrows \mathcal{G}_R : H_P$$

then $\mathcal{C}_A = Im(H_P)$ and $\mathcal{G}_R = Im(T_P)$.

Theorem 2.4.1 [S2 , theorem 1.3 , 1979] *Let $_AP_R$ be a bimodule , $T_P = (- \otimes_A P)$ and $H_P = Hom_R(P_R, -)$ with the associated natural morphisms σ and ρ . Then the following statements are equivalent :*

(1) $T_P : Im(H_P) \to Im(T_P)$ *and* $H_P : Im(T_P) \to Im(H_P)$ *are inverse category equivalences .*

(2) $T_P : Copres(K_A) \to Pres(P_R)$ *and* $H_P : Pres(P_R) \to Copres(K_A)$ *are inverse category equivalences . Here* $K_A = Hom_R(P_R, Q_R)_A$ *,* Q_R *is an injective cogenerator in* $Mod\text{-}R$ *.*

(3) $\rho(T_P(-)) : T_P H_P T_P \to T_P$ *and* $\sigma(H_P(-)) : H_P T_P H_P \to H_P$ *are natural isomorphisms .*

(4) $\rho(T_P(-)) : T_P H_P T_P \to T_P$ *is a natural isomorphism .*

(5) $\sigma(H_P(-)) : H_P \to H_P T_P H_P$ *is a natural isomorphism .*

(6) $Coker(\sigma_N) \otimes_A P = 0$ *for every* $N_A \in Mod\text{-}A$ *.*

(7) $Hom_R(P, Ker(\rho_M)) = 0$ *for every* $M_R \in Mod\text{-}R$ *.*

(8) *The functor* $T_P H_P : Mod\text{-}R \to Im(T_P)$ *is a right adjoint functor of the inclusion functor* $J : Im(T_P) \to Mod\text{-}R$ *with adjoint isomorphism* $Hom_R(-, \rho(-))$ *.*

Proof :
For brevity , sometimes we may write $T = T_P$ and $H = H_P$.
(2) $\Rightarrow$ (1) $Pres(P_R) \supseteq Im(T_P)$ and $Copres(K_A) \supseteq Im(H_P)$ in general .

(1) $\Rightarrow$ (2) Now $\rho : T_P H_P \to Id_{Im(T_P)}$ and $\sigma : Id_{Im(H_P)} \to H_P T_P$ are natural isomorphisms . We need to show that $Pres(P_R) \subseteq Im(T_P)$ and $Copres(K_A) \subseteq Im(H_P)$.

For $Pres(P_R) \subseteq Im(T_P)$, see the proof of [theorem 2.3.1 (2) $\Rightarrow$ (3)] .

Let $N_A \in Copres(K_A)$ and so there is an exact sequence

$$0 \to N_A \to K_A^X \xrightarrow{g} K_A^Y$$

where X and Y are sets . Applying $T_P = (- \otimes_A P)$ we get an exact sequence

$$0 \longrightarrow Ker(g \otimes Id_P) \xrightarrow{i} K^X \otimes_A P \xrightarrow{g \otimes Id_P} K^Y \otimes_A P$$

where i = inclusion map . Since $K_A^X = (Hom_R(P,Q)^X \cong Hom_R(P,Q^X) \in Im(H_P)$ and similarly $K_A^Y \in Im(H_P)$. Then by applying H_P , we get a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_P(Ker(g \otimes Id_P)) & \xrightarrow{H_P(i)} & H_P(K^X \otimes_A P) & \xrightarrow{H_P(g \otimes Id_P)} & H_P(K^Y \otimes_A P) \\ & & h \uparrow & & \uparrow \cong & & \uparrow \cong \\ 0 & \longrightarrow & N_A & \longrightarrow & K_A^X & \xrightarrow{g} & K_A^Y \end{array}$$

and hence , by 5-lemma , $N_A \cong H_P(Ker(g \otimes Id_P)) \in Im(H_P)$.

(1) $\Leftrightarrow$ (3) By proposition 1.1.5 , (1) means that $\rho : T_P H_P \to Id_{Im(T_P)}$ and $\sigma : Id_{Im(H_P)} \to H_P T_P$ are natural isomorphisms . But $\rho : TH \to Id_{Im(T)}$ natural isomorphism iff $\rho(T(-)) : THT \to T$ natural isomprphism and $\sigma : HT \to Id_{Im(H)}$ natural isomorphism iff $\sigma(H(-)) : HTH \to H$ natural isomprphism . Note also that $\rho(T_P(-)) : T_P H_P T_P \to T_P$ and $\sigma(H_P(-)) : H_P \to H_P T_P H_P$ are natural morphisms .

(4) $\Leftrightarrow$ (6) Since $\rho(T(N)) \circ T(\sigma_N) = Id_{T(N)}$ for all $N \in Mod\text{-}A$.

$$\begin{array}{ccc} T(N) & & \\ T(\sigma_N) \downarrow & & \\ THT(N) & \xrightarrow{\rho(T(N))} & T(N) \end{array}$$

. So $\rho(T(N))$ is always epic and

$$THT(N) = Ker(\rho(T(N))) \bigoplus Im(T(\sigma_N)) \quad .$$

Since T_P is a right exact functor , it preserves cokerals . And then

$$\begin{aligned} Ker(\rho(T(N))) &\cong THT(N)/Im(T(\sigma_N)) \\ &= Coker(T(\sigma_N)) \\ &\cong T(Coker(\sigma_N)) \end{aligned}$$

. Thus , $\rho(T(N))$ is monic if and only if

$$Ker(\rho(T(N))) \cong T(Coker(\sigma_N)) = Coker(\sigma_N) \otimes_A P_R = 0 \quad .$$

(5) $\Leftrightarrow$ (7) $H(\rho_M) \circ \sigma(H(M)) = Id_{H(M)}$ for every $M \in Mod\text{-}R$. So $\sigma(H(M))$ is always monic and

$$HTH(M) = Ker(H(\rho_M)) \bigoplus Im(\sigma(H(M))) \ .$$

Because H is a left exact sequence , it preserves kernels and

$$\begin{aligned} Coker(\sigma(H(M))) &= HTH(M)/Im(\sigma(H(M))) \ . \\ &\cong Ker(H(\rho_M)) \\ &\cong H(Ker(\rho_M)) \end{aligned}$$

. Thus , $\sigma(H(M))$ is epic if and only if

$$0 = Coker(\sigma(H(M))) \cong H(Ker(\rho_M)) = Hom_R(P, Ker(\rho_M)) \ .$$

(3) $\Rightarrow$ (4) and (3) $\Rightarrow$ (5) are clear .

(5) $\Rightarrow$ (3) Note that (5) $\Leftrightarrow$ (7) . Now It suffices to prove that $\rho(T(N))$ is monic for each $N \in Mod\text{-}A$. Since

$$Ker(\rho(T(N))) \cong T(Coker(\sigma_N)) \in Im(T_P) \subseteq Gen(P_R)$$

, so the natural homomorphism

$$Hom_R(P, Ker(\rho(T(N)))) \otimes_A P \to Ker(\rho(T(N)))$$

is epic . But , by condition (7) , $Hom_R(P, Ker(\rho(T(N)))) = 0$. Hence $Ker(\rho(T(N))) = 0$.

(4) $\Rightarrow$ (3) Note that (4) $\Leftrightarrow$ (6) . Now It suffices to prove that $\sigma(H(M))$ is epic for every $M \in Mod\text{-}R$ i.e. we need to show that $Coker(\sigma(H(M))) = 0$.
Since $H(Ker(\rho_M)) \cong Coker(\sigma(H(M)))$ and by condition (6) ,

$$T_P(H(Ker(\rho_M))) \cong T_P(Coker(\sigma(H(M)))) = 0$$

i.e. $Hom_R(P, Ker(\rho_M)) \otimes_A P = 0$ and this implies that $Hom_R(P, Ker(\rho_M)) = 0$ because $[f \otimes p \mapsto f(p) = 0]$ for every $f \in Hom_R(P, Ker(\rho_M))$ and every $p \in P$. But then

$$0 = Hom_R(P, Ker(\rho_M)) \cong Coker(\sigma(H(M)))$$

(1) $\Rightarrow$ (8) Now we have $T : Im(H) \to Im(T)$ is full and faithful i.e.

$$T : Hom_A(N_1, N_2) \to Hom_R(TN_1, TN_2)$$

is a $\mathbb{Z}$-isomorphism for N_1 , $N_2 \in Mod\text{-}A$.

We need to prove that (J, TH) is an adjoint pair with respect to categories $Im(T)$ and $Mod\text{-}R$ with adjoint isomorphism $Hom_R(-, \rho(-))$.

Let $K \in Im(T)$ and $M \in Mod\text{-}R$, then

$$\begin{array}{rcl} Hom_R(K, TH(M)) & \overset{\rho_K^*}{\cong} & Hom_R(TH(K), TH(M)) \\ & \overset{T^{-1}}{\cong} & Hom_A(H(K), H(M)) \\ & \overset{\Phi}{\cong} & Hom_R(TH(K), M) \\ & \overset{(\rho_K^{-1})^*}{\cong} & Hom_R(K, M) \\ & = & Hom_R(J(K), M) \end{array}$$

where $\rho_K^* = Hom_R(\rho_K, TH(M))$ and $(\rho_K^{-1})^* = Hom_R(\rho_K^{-1}, M)$. Let $(\rho_M)_* = Hom_R(K, \rho(M))$. We want

$$(\rho_K^{-1})^* \circ \Phi \circ T^{-1} \circ \rho_K^* = Hom_R(K, \rho(M))$$

but this is equivalent to

$$(\rho_K^{-1})^* \circ \Phi = (\rho_M)_* \circ (\rho_K^*)^{-1} \circ T \quad .$$

Note that both sides are maps : $Hom_A(HK, HM) \to Hom_R(K, M)$ and clearly $(\rho_K^*)^{-1} = (\rho_K^{-1})^*$. Now let $h \in Hom_A(HK, HM)$,

$$\begin{array}{rcl} LHS & = & ((\rho_K^{-1})^* \circ \Phi)(h) \\ & = & \Phi(h) \circ (\rho_K^{-1}) \\ RHS & = & ((\rho_M)_* \circ (\rho_K^*)^{-1} \circ T)(h) \\ & = & ((\rho_M)_* \circ (\rho_K^{-1})^* \circ T)(h) \\ & = & \rho_M \circ T(h) \circ \rho_K^{-1} \end{array}$$

, so it suffices to check $\Phi(h) = \rho_M \circ T(h) : THK \to M$, let $f \in H(K)$ and $p \in P$,

$$\begin{array}{rcl} (\rho_M \circ T(h))(f \otimes p) & = & (\rho_M \circ h \otimes Id_P)(f \otimes p) \\ & = & \rho_M(h(f) \otimes p) \\ & = & h(f)(p) \\ & = & \Phi(h)(f \otimes p) \end{array}$$

(8) $\Rightarrow$ (5) Since $\sigma(H(-))$ is a natural morphism , it suffices to show that $\sigma(H(L)) : H(L) \to HTH(L)$ is an isomorphism for every $L \in Mod\text{-}R$.

Clearly $A \otimes_A P \in Im(T)$, by condition (8) ,

$$(\rho(L))_* = Hom_R(A \otimes_A P, \rho(L)) : Hom_R(A \otimes_A P, TH(L)) \to Hom_R(A \otimes_A P, L)$$

is an isomorphism . Consider

$$\begin{array}{rcl} H(L) = Hom_R(P, L) & \overset{\mu^*}{\cong} & Hom_R(A \otimes P, L) \\ & \overset{((\rho(L))_*)^{-1}}{\cong} & Hom_R(A \otimes P, THL) \\ & \overset{(\Phi)^{-1}}{\cong} & Hom_A(A, HTHL) \\ & \overset{v}{\cong} & HTHL \\ & = & Hom_R(P, (HL) \otimes_A P) \end{array}$$

where $\mu : A \otimes_A P \to P$ $[a \otimes p \mapsto ap]$ and $\mu^* = Hom_R(\mu, L)$, $v(h) = h(1)$ for every $h \in Hom_A(A, HTHL)$ and $1 \in A$. Now we check that

$$v \circ (\Phi)^{-1} \circ ((\rho(L))_*)^{-1} \circ \mu^* = \sigma(H(L)) : H(L) \to HTH(L)$$

i.e. for every $f \in H(L)$, we want

$$(v \circ (\Phi)^{-1} \circ ((\rho(L))_*)^{-1} \circ \mu^*)(f) = \sigma(H(L))(f) : P \to (HL) \otimes_A P \quad .$$

Note that $((\rho(L))_*)^{-1} = (\rho(L)^{-1})_*$ and then we have

$$\begin{array}{rcl} (v \circ (\Phi)^{-1} \circ (\rho(L)^{-1})_* \circ \mu^*)(f) & = & v(\Phi^{-1}(\rho(L)^{-1}(f \circ \mu)) \\ & = & (\Phi^{-1}(\rho(L)^{-1}(f \circ \mu)(1) \end{array}$$

. Thus , for every $p \in P$,

$$\begin{array}{rcl} (\Phi^{-1}(\rho(L)^{-1}(f \circ \mu)(1)(p) & = & \rho(L)^{-1}(f \circ \mu)(1 \otimes p) \\ & = & \rho(L)^{-1}(f(p)) \\ & = & f \otimes p \\ & = & \sigma(H(L))(f)(p) \end{array}$$

$\square$

Chapter 3

$*$-modules and Tilting Modules

3.1 The Equivalence $Cogen(K_A) \sim Gen(P_R)$

Let ${}_AP_R$ be a bimodule , $T_P = (- \otimes_A P)$ and $H_P = Hom_R(P, -)$. Recall that if we choose a cogenerator Q_R of $Mod\text{-}R$ and let $K_A = Hom_R(P, Q)$ then we can define a category $Cogen(K_A)$. However it should be noted that **proposition 1.1.4 (2)** and **(3)** hold regardless of which cogenerator Q_R we choose i.e. for any cogenerator $Q_R \in Mod\text{-}R$ with $K = Hom_R(P, Q)$, $Im(H_P) \subseteq Copres(K_A)$ and σ_N is monic iff $N \in Cogen(K_A)$.

Also by **proposition 1.1.4 (3)** and **(4)** we know that $Cogen(K_A)$ and $Gen(P_R)$ are the largest subcategories between which the pair of functor (T_P, H_P) may induce an equivalence i.e. if we have an equivalence

$$T_P : \mathcal{C}_A \rightleftarrows \mathcal{G}_R : H_P$$

then $\mathcal{C}_A \subseteq Cogen(K_A)$ and $\mathcal{G}_R \subseteq Gen(P_R)$.

Clearly if $A = End(P_R)$ then $A \in Im(H_P) \subseteq Cogen(K_A)$. Moreover $Cogen(K_A)$ is closed under taking submodules (and arbitrary direct products) and $Gen(P_R)$ is closed under taking epimorphic images and arbitrary direct sums . Menini and Orsatti [MO , 1989] studied the equivalences between such two categories and also obtained a representation theorem .

Theorem 3.1.1 [MO , theorem 3.1] *Let A , R be rings , Q_R an arbitrary cogenerator of $Mod\text{-}R$. If there are given :*

(i) $\mathcal{C}_A \subseteq Mod\text{-}A$ *a full subcategory , closed under taking submodules and such that* $A_A \in \mathcal{C}_A$ *;*

(ii) $\mathcal{G}_R \subseteq Mod\text{-}R$ *a full subcategory , closed under taking epimorphic images and arbitrary direct sums ;*

and a category equivalence

$$T : \mathcal{C}_A \rightleftarrows \mathcal{G}_R : H$$

with T *,* H *additive functors . Then there exists a bimodule* ${}_AP_R$ *, unique up to isomorphism , with the following properties :*

(1) $P_R \in \mathcal{G}_R$ *,* $A \cong End(P_R)$ *cononically .*

(2) $T \cong (- \otimes_A P)$ *and* $H \cong Hom_R(P, -)$ *naturally .*

(3) $\mathcal{G}_R = Gen(P_R)$ *and* $\mathcal{C}_A = Cogen(K_A)$ *, where* $K_A = Hom_R(P_R, Q_R)$ *.*

Proof :
Note that now (T, H) is an adjoint pair with respect to categories $\mathcal{C}_A$ and $\mathcal{G}_R$. Also T and H are full and faithful .
(1) Let $P = T(A)_R \in \mathcal{G}_R$. Since $A = {}_AA_A$, so P has a canonical bimodule structure ${}_AP_R$ defined by the composition of the ring homomorphisms

$$A \xrightarrow{cano.} End(A_A) \xrightarrow{T} End(T(A)_R) = End(P_R)$$

. But T is full and faithful and hence $A \cong End(P_R)$ canonically . [AF , 20.3]

(2) For every $M \in \mathcal{G}_R$, there are isomorphisms

$$H(M) \cong Hom_A(A, H(M)) \cong Hom_R(T(A), M) = Hom_R(P, M)$$

which are natural in $M \in \mathcal{G}_R$. So $H \cong Hom_R(P, -)$.

Define $T_P = (- \otimes_A P)$, then $Im(T_P) \subseteq Gen(P_R) \subseteq \mathcal{G}_R$ because $P_R \in \mathcal{G}_R$ which is closed under direct sums and quotient modules . Now we have that $T_P : \mathcal{C}_A \longrightarrow \mathcal{G}_R$ and T are both left adjoints of $H \cong H_P$ and hence , by the uniqueness of the left adjoint of H , $T \cong (- \otimes_A P) = T_P$.

(3) Now $\mathcal{G}_R = Im(T) = Im(T_P) \subseteq Gen(P_R) \subseteq \mathcal{G}_R$ and so $\mathcal{G}_R = Gen(P_R)$.

On the other hand , $\mathcal{C}_A = Im(H) = Im(H_P) \subseteq Cogen(K_A)$. Note that

$$K_A = Hom_R(P, Q) = Hom_R(P, t_P(Q))$$

and $t_P(Q) \in Gen(P_R)$, so $K_A \in \mathcal{C}_A$. By assumption $\mathcal{C}_A$ is closed under submodules . Now we claim that $\mathcal{C}_A$ is closed under taking arbitrary products so that $Cogen(K_A) \subseteq \mathcal{C}_A$.

To prove this , let $\{N_i\}_{i \in I}$ be a family in $\mathcal{C}_A$ and hence there exists $M_i \in \mathcal{G}_R$ such that $Hom_R(P, M_i) = H_P(M_i) = N_i$ for every $i \in I$. Since $t_P(\prod_{i \in I} M_i) \in Gen(P_R) = \mathcal{G}_R$ so $Hom_R(P, t_P(\prod_{i \in I} M_i)) \in Im(H_P) = \mathcal{C}_A$. But

$$Hom_R(P, t_P(\prod_{i \in I} M_i)) = Hom_R(P, \prod_{i \in I} M_i)) \cong \prod_{i \in I} Hom_R(P, M_i) = \prod_{i \in I} N_i$$

which is in $\mathcal{C}_A$.

Finally if there is some P_R satisfying conditions (1) to (3) , then $A \cong End(P_R) \cong H(P_R)$ and so $T(A) \cong TH(P_R) \cong P_R$ i.e such a P_R must be isomorphic to $T(A)$. □

Definition 3.1.2 *Let P_R be a module , $A = End(P_R)$, $T_P = (- \otimes_A P)$ and $H_P = Hom_R(P, -)$. If (T_P, H_P) induce an equivalence between $Cogen(K_A)$ and $Gen(P_R)$, where $K_A = Hom_R(P, Q_R)$ and Q_R is a fixed , but arbitrary , cogenerator in Mod-R , then we call P_R a $*$-module .*

Clearly $Cogen(K_A) \sim Gen(P_R)$ is a further generalization of the equivalence studied by Fuller (see theorem 2.2.2) . In [MO] , Menini and Orsatti proved that every *tilting module* (see definition 3.4.1) is a $*$-module and pointed out that there are tilting modules which are not quasi-progenerators . Hence this generalization is not trivial . Moreover , naturally , they also posed a question on the characterizations of $*$-modules . We are going to solve these problems .

Now we first give some useful characterizations of $*$-modules due to Colpi [C1 , 1990] . We begin with the following

Definition 3.1.3

(1) $P_R \in Mod\text{-}R$ *is selfsmall if , for any set* X *, the canonical map*

$$\eta : Hom_R(P,P)^{(X)} \longrightarrow Hom_R(P,P^{(X)}) \quad [(g_x)_{x\in X} \longmapsto [p \mapsto (g_x(p))_{x\in X}]]$$

is an isomorphism .

Note that η is always monic . Let $\pi_x : P^{(X)} \longrightarrow P$ be the x-th canonical projection , then P_R is selfsmall iff for any $g \in Hom_R(P,P)^{(X)}$, $\pi_x \circ g \neq 0$ for only finitely many $x \in X$. So every finitely generated module is selfsmall but not conversely .

(2) $P_R \in Mod\text{-}R$ *is w-*$\sum$*-quasi-projective if for any* P_R*-presentation of* M_R

$$(ex) : \quad P_R^{(X)} \longrightarrow P_R^{(Y)} \xrightarrow{\varphi} M_R \longrightarrow 0$$

and for any $f \in Hom_R(P,M)$ *, there exists a* $g \in Hom_R(P,P_R^{(Y)})$ *such that* $\varphi \circ g = f$

$$\begin{array}{ccccccc} P_R^{(X)} & \longrightarrow & P_R^{(Y)} & \xrightarrow{\varphi} & M_R & \longrightarrow & 0 \\ & & {\scriptstyle \exists\, g}\uparrow & & \uparrow{\scriptstyle f} & & \\ & & P_R & \xrightarrow{=} & P_R & & \end{array}$$

i.e. $Hom_R(P,P^{(X)}) \xrightarrow{Hom_R(P,\varphi)} Hom_R(P,M)$ *is epic .*

However , $Hom_R(P,-)$ may not preserve the exactness of (ex) and so this is a weaker notion of semi-$\sum$-quasi-projective module (see definition 2.2.4) .

Before we study the equivalence $Cogen(K_A) \sim Gen(P_R)$, we consider the case $Cogen(K_A) \sim Pres(P_R)$ first and the results are also due to Colpi [C1 , 1990] .

Proposition 3.1.4 [C1 , proposition 3.7] *Let* $P_R \in Mod\text{-}R$ *,* $A = End(P_R)$ *,* Q_R *be a cogenerator of* $Mod\text{-}R$ *and let* $K_A = Hom_R(P,Q)$ *. Then the following conditions are equivalent :*

(1) T_P *and* H_P *define an equivalence between* $Cogen(K_A)$ *and* $Pres(P_R)$ *.*

(2) *Every module of* $Cogen(K_A)$ *is reflexive i.e.* $\sigma(N_A)$ *is an isomorphism for every* $N_A \in Cogen(K_A)$ *.*

(3) *For any* $N \in Mod\text{-}A$, σ_N *is an epimorphism .*

(4) P_R *is selfsmall and* $H_P T_P$ *preserves the epimorphisms of* $Mod\text{-}A$ *.*

(5) P_R *is selfsmall and w-*$\sum$*-quasi-projective .*

(6) *For any exact sequence in* $Mod\text{-}R$ *:*

$$P_R^{(X)} \longrightarrow P_R^{(Y)} \xrightarrow{\varphi} M_R \longrightarrow 0$$

and if we write $\varphi = (\varphi_y)_{y \in Y}$ *for* $\varphi_y \in Hom_R(P, M)$ *, we have*

$$Hom_R(P, M)_A = \sum_{y \in Y} \varphi_y A$$

.

Note that $Cogen(K_A) \sim Pres(P_R)$ if and only if $Cogen(K_A) \sim Im(T_P)$ by theorem 2.4.1 .

Proof :
(1) $\Rightarrow$ (2) Clear by proposition 1.1.5 .
(2) $\Rightarrow$ (3) [MO , proposition 3.7] Let $N \in Mod\text{-}A$ and $\sigma_N : N \longrightarrow H_P T_P(N)$. So

$$N/Ker(\sigma_N) \leq H_P T_P(N) \in Im(H_P) \subseteq Cogen(K_A)$$

and hence $N/Ker(\sigma_N)$ is an object of $Cogen(K_A)$ and $\sigma_{N/Ker(\sigma_N)}$ is an isomorphism by assumption (2) .

Next we prove that $T_P(N/Ker(\sigma_N)) \cong T_P(N)$. Consider the exact sequence

$$0 \longrightarrow Ker(\sigma_N) \xrightarrow{i} N \xrightarrow{\pi} N/Ker(\sigma_N) \longrightarrow 0$$

. Applying the right exact functor $(- \otimes_A P) = T_P$ we have

$$Ker(\sigma_N) \otimes_A P \xrightarrow{i \otimes Id_P} N \otimes_A P \xrightarrow{\pi \otimes Id_P} (N/Ker(\sigma_N)) \otimes_A P \longrightarrow 0$$

and $Ker(\pi \otimes Id_P) = Im(i \otimes Id_P) = (i \otimes Id_P)(Ker(\sigma_N) \otimes_A P) = 0$ because for any $k \in Ker(\sigma_N)$ and $p \in P$,

$$(i \otimes Id_P)(k \otimes p) = k \otimes p = \sigma_N(k)(p) = 0$$

. So $T_P(\pi) = \pi \otimes Id_P$ is an isomorphism i.e. $T_P(N/Ker(\sigma_N)) \cong T_P(N)$. Now we have the commutative diagram with exact rows

$$\begin{array}{ccccc} N & \xrightarrow{\pi} & N/Ker(\sigma_N) & \longrightarrow & 0 \\ \sigma_N \downarrow & & \downarrow \sigma_{(N/Ker(\sigma_N))} & & \\ H_PT_P(N) & \xrightarrow{H_PT_P(\pi)} & H_PT_P(N/Ker(\sigma_N)) & & \end{array}$$

and hence , as $H_PT_P(\pi)$ and $\sigma_{N/Ker(\sigma_N)}$ are isomorphisms , σ_N is epic .

(3) $\Rightarrow$ (1) For every $N \in Cogen(K_A)$, by assumption (3) , σ_N is an isomorphism .

Now we have ρ_W is an isomorphism for every $W \in Im(T_P)$ because if we write $W = T_P(N)$ then $\rho_{T_P(N)} \circ T_P(\sigma_N) = Id_{T_P(N)}$ and σ_N is an isomorphism .

Therefore we have the equivalence $Cogen(K_A) \sim Im(T_P)$ and hence the equivalence $Cogen(K_A) \sim Pres(P_R)$ by theorem 2.4.1 .

(3) $\Rightarrow$ (4)
(i) $A = Hom_R(P,P) = H_P(P_R) \in Im(H_P) \subseteq Cogen(K_A)$ so $A^{(X)} \in Cogen(K_A)$ and by (3) $\Rightarrow$ (1) , we have $\sigma(A^{(X)}) : A^{(X)} \cong H_PT_P(A^{(X)})$. Then

$$\begin{aligned} (Hom_R(P,P))^{(X)} = A^{(X)} & \overset{\sigma(A^{(X)})}{\cong} H_PT_P(A^{(X)}) \\ & = Hom_R(P, A^{(X)} \otimes_A P) \\ & \overset{\phi_*}{\cong} Hom_R(P, P^{(X)}) \end{aligned}$$

where $\phi : A^{(X)} \otimes_A P \longrightarrow P^{(X)}$ is defined by $\phi((a_x)_{x\in X} \otimes p) = (a_x(p))_{x\in X}$ for $(a_x)_{x\in X} \in A^{(X)}$ and $p \in P$ and $\phi_* = Hom_R(P, \phi)$. It is clear that the composition map

$$\phi_* \circ \sigma(A^{(X)}) : Hom_R(P,P)^{(X)} = A^{(X)} \longrightarrow Hom_R(P, P^{(X)})$$

is canonical and so P_R is selfsmall .
(ii) Let $N \xrightarrow{f} L \longrightarrow 0$ be an epimorphism in $Mod\text{-}A$. Then from the commu-

tative diagram

$$\begin{array}{ccccc} N & \xrightarrow{f} & L & \longrightarrow & 0 \\ {\scriptstyle \sigma_N}\downarrow & & \downarrow{\scriptstyle \sigma_L\ epic} & & \\ H_PT_P(N) & \xrightarrow{H_PT_P(f)} & H_PT_P(L) & & \end{array}$$

, $H_PT_P(f)$ is epic .

(4) $\Rightarrow$ (5) We need to show that P_R is w-$\sum$-quasi-projective i.e. for any exact sequence

$$P_R^{(X)} \xrightarrow{\psi} P_R^{(Y)} \xrightarrow{\varphi} M_R \longrightarrow 0$$

in Mod-R , $H_P(\varphi)$ is epic .

Write $\psi = (\psi_x)_{x\in X}$ and $\varphi = (\varphi_y)_{y\in Y}$ for

$$\psi_x \in Hom_R(P, P^{(Y)}) \underset{\cong}{\xleftarrow{\eta}} Hom_R(P,P)^{(Y)} = A^{(Y)}$$

where η is the canonical isomorphism as P_R is selfsmall . So for every $x \in X$, there exists $g_x \in Hom_R(P,P)^{(Y)} = A^{(Y)}$ such that $\psi_x = \eta(g_x)$. Define

$$K_A = \langle\ \psi_x \mid x \in X\ \rangle_A = \langle\ \eta(g_x) \mid x \in X\ \rangle_A \leq Hom_R(P, P^{(Y)})_A\ .$$

Note that

$$Im(\psi) = \sum_{x\in X} Im(\psi_x) = \sum_{x\in X} \eta(g_x)(P) = KP \leq P^{(Y)}$$

and then we have $Im(\psi) = Ker(\varphi) = KP$ and the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & KP & \longrightarrow & P^{(Y)} & \xrightarrow{\varphi} & M & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle =} & & \uparrow{\scriptstyle =} & & \uparrow{\scriptstyle \overline{\varphi}\ iso} & & \\ 0 & \longrightarrow & KP & \longrightarrow & P^{(Y)} & \xrightarrow{\mu} & P^{(Y)}/KP & \longrightarrow & 0 \end{array}$$

where $\varphi = \overline{\varphi}\circ\mu$ for $\mu =$ natural epimorphism and $\overline{\varphi} =$ canonical isomorphism by first isomorphism theorem [AF , 3.7] .

Let $N_A = \langle\, g_x \mid x \in X \,\rangle_A \leq (Hom_R(P,P))^{(Y)} = A^{(Y)}$ and then $\eta(N) = K$. Consider the exact sequence

$$0 \to N \xrightarrow{i} A^{(Y)} \xrightarrow{\pi} A^{(Y)}/N \to 0$$

and applyiny T_P we get the commutative diagram with exact rows

$$\begin{tikzcd}
 & N \otimes_A P \arrow[r, "i\otimes Id_P"] \arrow[d, "\exists\, f"] & A^{(Y)} \otimes_A P \arrow[r, "\pi\otimes Id_P"] \arrow[d, "\phi"] & (A^{(Y)}/N) \otimes_A P \arrow[r] \arrow[d, "\exists\, h"] & 0 \\
0 \arrow[r] & KP \arrow[r] & P^{(Y)} \arrow[r, "\mu"] & P^{(Y)}/KP \arrow[r] & 0
\end{tikzcd}$$

where

$$\phi : A^{(Y)} \otimes_A P \cong P^{(Y)} \quad [(a_y)_{y\in Y} \otimes p \mapsto (a_y p)_{y\in Y}]$$

is the canonical isomorphism .

We check that $KP \supseteq Im(\phi \circ (i \otimes Id_P))$. Note that

$$N_A = \langle\, g_x \mid x \in X \,\rangle_A \leq A^{(Y)}$$

, so $\{g_x \otimes p \mid x \in X \text{ and } p \in P\}$ is a set of generators of $N \otimes_A P$. Write $g_x = (a_y^x)_{y\in Y} \in N$, then

$$\phi((a_y^x)_{y\in Y} \otimes p) = (a_y^x p)_{y\in Y} = \eta((a_y^x))(p) = \eta(g_x)(p) \in KP$$

. Moreover , $Ker(\pi \otimes Id_P) \subseteq Ker(\mu \circ \phi)$ since $Ker(\pi \otimes Id_P) = Im(i \otimes Id_P)$ and

$$\mu \circ \phi(Im(i \otimes Id_P)) \subseteq \mu(KP) = \overline{0}$$

. Therefore , by [AF , 3.6] , there are morphisms f and h which complete the commutative diagram and from this diagram , h is an isomorphism .

Then $\mu = h \circ \pi \otimes Id_P \circ \phi^{-1}$ and hence $\varphi = \overline{\varphi} \circ h \circ \pi \otimes Id_P \circ \phi^{-1}$. Finally we have $H_P(\varphi) = H_P(\overline{\varphi}) \circ H_P(h) \circ H_P(\pi \otimes Id_P) \circ H_P(\phi^{-1})$ which is epic since H_P preserves isomorphisms and $H_P(\pi \otimes Id_P) = H_P T_P(\pi)$ is epic by assumption (4) .

(5) $\Rightarrow$ (6) For any exact sequence in *Mod-R* ,

$$P_R^{(X)} \to P_R^{(Y)} \xrightarrow{\varphi} M_R \to 0 \ .$$

Here we write $\varphi = (\varphi_y)_{y\in Y}$ for $\varphi_y \in Hom_R(P,M)_A$. We need to prove that $Hom_R(P,M)_A = \sum_{y\in Y}\varphi_y A$.

Let $f \in Hom_R(P,M)_A$, by assumption (5) , there is a

$$g \in Hom_R(P,P^{(Y)}) \overset{\eta}{\underset{\cong}{\longleftarrow}} Hom_R(P,P)^{(Y)} = A^{(Y)}$$

such that $\varphi \circ g = f$ and $g = \eta((a_y)_{y\in Y})$ for some $(a_y)_{y\in Y} \in A^{(Y)}$. Now for every $p \in P$ we have

$$\begin{aligned} f(p) = \varphi(g(p)) &= \varphi(\eta((a_y)_{y\in Y})(p)) \\ &= \varphi((a_y p)_{y\in Y}) \\ &= \textstyle\sum_{y\in Y}\varphi_y(a_y p) \\ &= (\textstyle\sum_{y\in Y}\varphi_y a_y)(p) \end{aligned}$$

, hence $f = \sum_{y\in Y}\varphi_y a_y \in \sum_{y\in Y}\varphi_y A$.

(6) $\Rightarrow$ (3) Consider the exact sequence ,

$$P_R^{(X)} \xrightarrow{0} P_R^{(X)} \xrightarrow{\varphi=id} P_R^{(X)} \longrightarrow 0$$

, write $id = (id_x)_{x\in X}$ for $id_x \in Hom_R(P,P^{(X)})$ and for every $f \in Hom_R(P,P^{(X)})$, by assumption (6) , there exists $a_x \in A$ such that $f(p) = (\sum_{x\in X} id_x a_x)(p)$ for every $p \in P$. So $f = id((a_x)_{x\in X})$ and hence $Hom_R(P,P)^{(X)} \cong Hom_R(P,P^{(X)})$ canonically i.e. P_R is selfsmall .

Now let $N \in Mod\text{-}A$ and

$$A^{(X)} \longrightarrow A^{(Y)} \xrightarrow{\varphi} N \longrightarrow 0$$

be exact in $Mod\text{-}A$. Applying T_P and then we have the commutative diagram with exact rows

$$\begin{array}{ccccccc} A^{(X)} \otimes_A P & \longrightarrow & A^{(Y)} \otimes_A P & \xrightarrow{T_P(\varphi)} & T_P(N) & \longrightarrow & 0 \\ \cong\big\downarrow & & \cong\big\downarrow\phi & & =\big\downarrow & & \\ P^{(X)} & \longrightarrow & P^{(Y)} & \xrightarrow{h} & T_P(N) & \longrightarrow & 0 \end{array}$$

where $h = T_P(\varphi)\circ\phi^{-1}$ and we write $h = (h_y)_{y\in Y}$. Now we claim that $H_P(h)$ is epic and hence $H_P T_P(\varphi)$ is epic . To this end , let $g \in Hom_R(P, T_P(N)) = H_P T_P(N)$, by assumption (6) , we have

$$g = \sum_{y\in Y} h_y a_y = h \circ \bigoplus_{y\in Y} a_y$$

for some $a_y \in A$ and where $\bigoplus_{y\in Y} a_y : P \longrightarrow P^{(Y)}$ $[p \mapsto (a_y p)_{y\in Y}]$. Note here that only finitely many $a_y \neq 0$.

So $H_P(h)$ is epic and hence $H_P T_P(\varphi)$ is epic . Finally we consider the commutative diagram with exact rows

$$\begin{array}{ccccc} A^{(Y)} & \xrightarrow{\varphi} & N & \longrightarrow & 0 \\ \cong \Big\downarrow \sigma_{A^{(Y)}} & & \Big\downarrow \sigma_N & & \\ H_P T_P(A^{(Y)}) & \xrightarrow{H_P T_P(\varphi)} & H_P T_P(N) & \longrightarrow & 0 \end{array}$$

where $\sigma(A^{(Y)})$ is an isomorphism because P_R is selfsmall and hence σ_N is epic . $\square$

Now we can give some characterizations of a $*$-module P_R .

Theorem 3.1.5 ([C1, theorem 4.1 , 1990] and [MO , 1989]) *Let $P_R \in$ Mod-R , $A = End(P_R)$, Q_R be a cogenerator of Mod-R and let $K_A = Hom_R(P, Q)$. Then the following conditions are equivalent :*

(1) *P_R is a $*$-module i.e. T_P and H_P define an equivalence between $Cogen(K_A)$ and $Gen(P_R)$.*

(2) *For any $N \in$ Mod-A , σ_N is an epimorphism and for any $M \in$ Mod-R , ρ_M is a monomorphism .*

(3) *$Gen(P_R) = Pres(P_R)$, P_R is selfsmall and w-$\sum$-quasi-projective .*

(4) *For any $M \in Gen(P_R)$ and any exact sequence*

$$(ex): \; 0 \longrightarrow Ker(\varphi) \longrightarrow P_R^{(X)} \xrightarrow{\varphi} M \longrightarrow 0$$

and if we write $\varphi = (\varphi_x)_{x\in X}$ for $\varphi_x \in Hom_R(P, M)_A$, then the following conditions are equivalent :

(a) *$Hom_R(P, M) = \sum_{x\in X} \varphi_x A$.*

(b) *$Ker(\varphi) \in Gen(P_R)$ i.e. (ex) is obtain from a P-presentation of M .*

(5) *P_R is selfsmall and for any set X , if $0 \longrightarrow M \xrightarrow{i} P_R^{(X)}$ is an exact sequence then*
$M \in Gen(P_R)$ if and only if $Ext_R^1(P,i) : Ext_R^1(P,M) \longrightarrow Ext_R^1(P,P_R^{(X)})$ is monic .

Proof :
(2) $\Rightarrow$ (1) Now we have that for any $N \in Cogen(K_A)$, σ_N is an isomorphism and for any $M \in Gen(P_R)$, ρ_M is an isomorphism .

(1) $\Rightarrow$ (2) By proposition 3.1.4 , because now T_P and H_P subordinate an equivalence between $Cogen(K_A)$ and $Pres(P_R)$ ($\subseteq Gen(P_R)$) , σ_N is epic for any $N \in Mod\text{-}A$.

On the other hand , for $M \in Mod\text{-}R$, note that $t_P(M) \in Gen(P_R)$ and $H_P(M) = H_P(t_P(M))$. We consider the commutative diagram

$$\begin{array}{ccc} t_P(M) & \xrightarrow{inc.} & M \\ {\scriptstyle \rho_{t_P(M)}}\Big\uparrow{\scriptstyle \cong} & & \Big\uparrow{\scriptstyle \rho_M} \\ T_PH_P(t_P(M)) & \xrightarrow{=} & T_PH_P(M) \end{array}$$

, by condition (1) , $\rho_{t_P(M)}$ is an isomorphism and so ρ_M is monic .

(1) $\Rightarrow$ (3) Now $Im(T_P) = Gen(P_R)$, so $Pres(P_R) = Gen(P_R)$ and (3) follows from proposition 3.1.4 .
(3) $\Rightarrow$ (1) By proposition 3.1.4 .

(3) $\Rightarrow$ (5) Now suppose that an exact sequence

$$(se) : \quad 0 \longrightarrow M \xrightarrow{i} P_R^{(X)} \xrightarrow{\pi} P_R^{(X)}/M \longrightarrow 0$$

is given .
(i) Let $M \in Gen(P_R)$ and so we have an exact sequence $P_R^{(I)} \longrightarrow M \longrightarrow 0$ for some set I . We need to prove that $Ext_R^1(P,i)$ is monic . First applying H_P to (se) , we get the long exact sequence [R , 5.2.28] ,

$$0 \longrightarrow H_P(M) \xrightarrow{H_P(i)} H_P(P^{(X)}) \xrightarrow{H_P(\pi)} H_P(P^{(X)}/M)$$

$$\xrightarrow{\delta} Ext_R^1(P,M) \xrightarrow{Ext_R^1(P,i)} Ext_R^1(P,P^{(X)})$$

. But $H_P(\pi)$ is epic because P_R is w-$\sum$-quasi-projective , so $\delta = 0$ and hence $Ker(Ext^1_R(P,i)) = 0$ i.e. $Ext^1_R(P,i)$ is monic .
(ii) If $0 \longrightarrow M \xrightarrow{i} P_R^{(X)}$ is exact and

$$Ext^1_R(P,M) \xrightarrow{Ext^1_R(P,i)} Ext^1_R(P,P^{(X)}/M)$$

is monic , then $\delta = 0$, so $H_P(\pi)$ is epic i.e. the sequence

$$0 \longrightarrow H_P(M) \xrightarrow{H_P(i)} H_P(P^{(X)}) \xrightarrow{H_P(\pi)} H_P(P^{(X)}/M) \longrightarrow 0$$

is exact . Also note that $P^{(X)}$, $P^{(X)}/M \in Gen(P_R)$ and because (3) implies (1) , $\rho(P^{(X)})$ and $\rho(P^{(X)}/M)$ are isomorphisms . Now we get the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "i"] & P_R^{(X)} \arrow[r, "\pi"] & P_R^{(X)}/M \arrow[r] & 0 \\
T_PH_P(M) \arrow[r] \arrow[u, "\rho_M"] & T_PH_P(P_R^{(X)}) \arrow[r, "T_PH_P(\pi)"] \arrow[u, "\rho", "\cong"'] & T_PH_P(P^{(X)}/M) \arrow[r] \arrow[u, "\rho", "\cong"'] & 0
\end{tikzcd}$$

and by [AF , lemma 3.14] , ρ_M is epic and hence $M \in Gen(P_R)$.

(5) $\Rightarrow$ (4) For any $M \in Gen(P_R)$ and any exact sequence

$$(ex): \ 0 \longrightarrow Ker(\varphi) \xrightarrow{i} P_R^{(X)} \xrightarrow{\varphi} M \longrightarrow 0$$

, we write $\varphi = (\varphi_x)_{x \in X}$ and applying H_P to (ex) , we get the long exact sequence

$$0 \longrightarrow H_P(Ker(\varphi)) \longrightarrow H_P(P^{(X)}) \xrightarrow{H_P(\varphi)} H_P(M)$$
$$\xrightarrow{\delta} Ext^1_R(P, Ker(\varphi)) \xrightarrow{Ext^1_R(P,i)} Ext^1_R(P, P^{(X)})$$

. Since , by condition (5) , P_R is selfsmall . Let

$$\eta : A^{(X)} = Hom_R(P,P)^{(X)} \cong Hom_R(P, P^{(X)})$$

be the canonical isomorphism . If $H_P((\varphi_x)_{x\in X})$ is epic and then for $f \in Hom_R(P,M)$, there exists a $h \in Hom(P, P^{(X)})$ such that $(\varphi_x) \circ h = f$. Write $h = \eta((a_x)_{x \in X})$ and for every $p \in P$,

$$f(p) = (\varphi_x) \circ \eta((a_x))(p) = (\varphi_x)(a_x p) = (\sum_{x\in X} \varphi_x a_x)(p)$$

i.e. $f \in \sum_{x\in X} \varphi_x A$ and hence $Hom_R(P,M) = \sum_{x\in X} \varphi_x A$. So

$$Hom_R(P,M) = \sum_{x\in X} \varphi_x A \quad \text{if and only if} \quad H_P((\varphi_x)_{x\in X}) \text{ is epic .}$$

Now we have

$$\begin{aligned} Hom_R(P,M) = \textstyle\sum_{x\in X} \varphi_x A &\iff H_P((\varphi_x)_{x\in X}) \ \ epic \\ &\iff \delta = 0 \\ &\iff Ext^1_R(P,i) \ \ monic \\ &\overset{(5)}{\iff} Ker(\varphi) \in Gen(P_R) \end{aligned}$$

and condition (4) follows .

(4) $\Rightarrow$ (3) By the implications (b) $\Rightarrow$ (a) of condition (4) and (6) $\Rightarrow$ (5) of proposition 3.1.4 , P_R is selfsmall and w-$\sum$-quasi-projective .

We need to check that $Gen(P_R) \subseteq Pres(P_R)$. Let $M \in Gen(P_R)$ and so

$$M = \sum\{Im(x) \mid x \in Hom_R(P,M) \overset{def}{=} X\}$$

. Now for every $x \in X = Hom_R(P,M)$, define $\varphi_x : P \longrightarrow M$ by $\varphi_x(p) = x(p)$ for $p \in P$. And so there is an exact sequence

$$0 \longrightarrow Ker(\varphi) \overset{i}{\longrightarrow} P_R^{(X)} \overset{\varphi}{\longrightarrow} M \longrightarrow 0$$

, here we write $\varphi = (\varphi_x)_{x\in X} = \bigoplus_{x\in X} \varphi_x$. Because for any $x_o \in Hom_R(P,M)$, we have that $\varphi_{x_o} = x_o : P \longrightarrow M$ and clearly $x_o \in \sum_{x\in X} \varphi_x A$ i.e. condition (a) of (4) is satisfied and so $Ker(\varphi) \in Gen(P_R)$. Thus , we get a P-presentation for M , hence $Gen(P_R) \subseteq Pres(P_R)$. $\square$

Corollary 3.1.6 [C1 , corollary 4.2] *Let P_R be a $*$-module , then $H_P = Hom_R(P,-)$ is an exact functor in $Gen(P_R)$, i.e. it preserves short exact sequences of $Gen(P_R)$.*

Proof :
Let $E \in Gen(P_R)$ and $J = Hom_R(P,E)$. For every $j \in J = Hom_R(P,E)$, we define $\varphi_j = j : P \longrightarrow E$. Then we have an epimorphism

$$\nu_E \overset{def}{=} \bigoplus_{j\in J} \varphi_j : P^{(J)} \longrightarrow E \longrightarrow 0 \quad [(p_j)_{j\in J} \mapsto \sum_{j\in J} \varphi_j p_j]$$

and by condition (4) of theorem 3.1.5 , $Ker(\nu_E) \in Gen(P_R)$.

Now let $0 \longrightarrow L \xrightarrow{i} M \xrightarrow{\pi} N \longrightarrow 0$ be an exact sequence in $Gen(P_R)$. We need to prove that $H_P(\pi)$ is epic .

Consider the commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
0 \longrightarrow & L \xrightarrow{i} & M & \xrightarrow{\pi} & N & \longrightarrow 0 & \\
 & & \nu_M \uparrow & & \nu_N \uparrow & & \\
 & & P^{(H_P(M))} & \underset{\exists\, h}{\longrightarrow} & P^{(H_P(N))} & & \\
 & & & & \uparrow & & \\
 & & & & P^{(Y)} & &
\end{array}$$

Since P_R is w-$\sum$-quasi-projective , so there exists h such that $\nu_N \circ h = \pi \circ \nu_M$. And we have the commutative diagram with exact rows

$$(CD) \qquad \begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{i_K} & P^{(H_P(M))} & \xrightarrow{\nu_N \circ h} & N & \longrightarrow & 0 \\
 & & \exists\, g \downarrow & & \downarrow \nu_M & & \downarrow = & & \\
0 & \longrightarrow & L & \xrightarrow{i} & M & \xrightarrow{\pi} & N & \longrightarrow & 0
\end{array}$$

where $K = Ker(\nu_N \circ h) = Ker(\pi \circ \nu_M) = \nu_M^{-1}(\pi^{-1}(0)) = \nu_M^{-1}(L)$ and $i_K =$ inclusion map . The existence of g is now clear [AF , 3.6] .

And then we apply H_P to (CD) and get the commutative diagram with exact rows

$$\begin{array}{ccccccc}
H_P(P^{(H_P(M))}) & \longrightarrow & H_P(N) & \xrightarrow{\delta} & Ext^1_R(P,K) & \xrightarrow{Ext^1_R(P,i_K)} & Ext^1_R(P,P^{(H_P(M))}) \\
\downarrow & & = \downarrow & & \downarrow & & \downarrow \\
H_P(M) & \xrightarrow{H_P(\pi)} & H_P(N) & \xrightarrow{\delta'} & Ext^1_R(P,L) & \longrightarrow & Ext^1_R(P,M)
\end{array}$$

Next we claim that $K \in Gen(P_R)$. To see this , consider the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & Ker(\nu_M) \arrow[r, "i_K"] & P^{(H_P(M))} \arrow[r, "\nu_M"] & M \arrow[r] & 0 \\
0 \arrow[r] & Ker(\nu_L) \arrow[r] & P^{(H_P(L))} \arrow[r, "\nu_L"] \arrow[u, "\exists\ \phi"'] & L \arrow[r] \arrow[u, "i"'] & 0
\end{tikzcd}$$

since $Ker(\nu_M) \in Gen(P_R)$ and P_R is w-$\sum$-quasi-prolective , there is a ϕ such that $\nu_M \circ \phi = i \circ \nu_L$ and

$$\begin{aligned}
\nu_M(Im(\phi) + Ker(\nu_M)) &= \nu_M \circ \phi(P^{(H_P(L))}) \\
&= i \circ \nu_L(P^{(H_P(L))}) \\
&= i(L) = L
\end{aligned}$$

. So $K = \nu_M^{-1}(L) = Im(\phi) + Ker(\nu_M) \in Gen(P_R)$ because $Im(\phi)$ and $Ker(\nu_M)$ are P_R-generated .

Finally by condition (5) of theorem 3.1.5 , as $0 \to K \xrightarrow{i_K} P^{(H_P(M))}$ and $K \in Gen(P_R)$, $Ext_R^1(P, i_K)$ is monic $\Rightarrow \delta = 0 \Rightarrow \delta' = 0 \Rightarrow H_P(\pi)$ is epic . So H_P preserves exact sequences in $Gen(P_R)$. □

Here we may refomulate the equivalence of conditions (1) and (5) of theorem 3.1.5 as

Proposition 3.1.7 [C1 , proposition 4.3] *Let $P_R \in Mod\text{-}R$, $A = End(P_R)$. The following conditions are equivalent :*

(1) *P_R is a $*$-module .*

(2) *P_R is selfsmall and for any exact sequence in $Mod\text{-}R$*

$$(ex): \quad 0 \to L \to M \xrightarrow{\pi} N \to 0$$

where $M \in Gen(P_R)$. Then $H_P(ex)$ is exact (i.e. $H_P(\pi)$ is epic) if and only if $L \in Gen(P_R)$.

Proof :
(2) $\Rightarrow$ (1) We verify the condition (5) of theorem 3.1.5 . Let

$$0 \to L \xrightarrow{i} P^{(X)} \xrightarrow{\pi} P^{(X)}/L \to 0$$

be exact and we have the long exact sequence

$$0 \to H_P(L) \xrightarrow{H_P(i)} H_P(P^{(X)}) \xrightarrow{H_P(\pi)} H_P(P^{(X)}/L)$$

$$\xrightarrow{\delta} Ext^1_R(P, L) \xrightarrow{Ext^1_R(P,i)} Ext^1_R(P, P^{(X)})$$

. Clearly $P_R^{(X)} \in Gen(P_R)$ and by condition (2) , $L \in Gen(P_R)$ iff $H_P(ex)$ is exact iff $H_P(\pi)$ is epic iff

$$Ext^1_R(P, i) : Ext^1_R(P, L) \to Ext^1_R(P, P_R^{(X)})$$

is monic .

(1) $\Rightarrow$ (2) If $L \in Gen(P_R)$ then $H_P(ex)$ is exact by corollary 3.1.6 .

Conversely if $H_P(ex)$ is exact then we have the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] & M \arrow[r] & N \arrow[r] & 0 \\
 & T_P H_P(L) \arrow[r] \arrow[u, "\rho_L"] & T_P H_P(M) \arrow[r] \arrow[u, "\rho_M", "\cong"'] & T_P H_P(N) \arrow[r] \arrow[u, "\rho_N", "\cong"'] & 0
\end{tikzcd}$$

where since M , $N \in Gen(P_R)$ and by assumption (1) , ρ_M and ρ_N are isomorphisms . Finally from the above diagram and by [AF , lemma 3.14] , ρ_L is epic i.e. $L \in Gen(P_R)$. □

We end this section by giving the following results which show the relations between $*$-modules and quasi-progenerators or progenerators .

Theorem 3.1.8 [C1 , theorem 4.7] *Let P_R be a $*$-module , $A = End(P_R)$, $H_P = Hom_R(P, -)$, $T_P = (- \otimes_A P)$ and $K_A = Hom_R(P, Q_R)$ for an arbitrary cogenerator $Q_R \in Mod\text{-}R$, so we have $Cogen(K_A) \sim Gen(P_R)$. Then*

(a) the following conditions are equivalent :

(1) P_R is a selfgenerator .

(2) P_R is quasi-projective .

(3) *$Gen(P_R)$ is closed under submodules i.e. $Gen(P_R) = \overline{Gen}(P_R)$ and we have the equivalence*

$$Cogen(K_A) \underset{H_P}{\overset{T_P}{\rightleftarrows}} \overline{Gen}(P_R) \ .$$

(4) *P_R is a quasi-progenerator i.e. $Mod\text{-}R \sim Gen(P_R)$.*

(b) the following conditions are equivalent :

(1) *P_R is a generator .*

(2) *P_R is a progenerator .*

Proof :
Part (a)
(1) $\Rightarrow$ (2) For any exact sequence

$$(ex1): \quad 0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

in $Mod\text{-}R$. By condition (1) , $K \in Gen(P_R)$ and then by proposition 3.1.7(2) , $H_P(ex1)$ is exact . Thus P_R is quasi-projective .

(2) $\Rightarrow$ (1) Let $K \leq P_R$ and consider the exact sequence

$$(ex2): \quad 0 \longrightarrow K \xrightarrow{i} P \longrightarrow P/K \longrightarrow 0$$

, by condition (2) , we get $H_P(ex2)$ is exact and so by proposition 3.1.7(2) , $K \in Gen(P_R)$.

(1) + (2) $\Rightarrow$ (3) Clear by lemma 2.2.14 or one may observe that now P is P^n-projective for $n \geq 1$.

(3) $\Rightarrow$ (4) Now $Gen(P_R) = \overline{Gen}(P_R)$ and we need to show that $Cogen(K_A) = Mod\text{-}A$.

To see this , let $N \in Mod\text{-}A$ and then we get an exact sequence

$$A^{(X)} \longrightarrow A^{(Y)} \longrightarrow N \longrightarrow 0$$

in $Mod\text{-}A$. Applying T_P , we have an exact sequence

$$P^{(X)} \longrightarrow P^{(Y)} \longrightarrow T_P(N) \longrightarrow 0$$

. By corollary 3.1.6 , the functor H_P is exact in $Gen(P_R) = \overline{Gen}(P_R)$. Then we have the commutative diagram with exact rows

$$\begin{array}{ccccccc}
A^{(X)} & \longrightarrow & A^{(Y)} & \longrightarrow & N & \longrightarrow & 0 \\
{\scriptstyle \sigma_{A^{(X)}}} \downarrow \cong & & \cong \downarrow {\scriptstyle \sigma_{A^{(Y)}}} & & \downarrow {\scriptstyle \sigma_N} & & \\
H_P(P^{(X)}) & \longrightarrow & H_P(P^{(Y)}) & \longrightarrow & H_P(T_P(N)) & \longrightarrow & 0
\end{array}$$

where $\sigma_{A^{(X)}}$ and $\sigma_{A^{(Y)}}$ are isomorphisms because $A = Hom_R(P,P) \in Im(H_P) \subseteq Cogen(K_A)$. Now from the diagram , we obtain that σ_N is an isomorphism for $N \in Mod\text{-}A$. Therefore , the equivalence is actually between $Mod\text{-}A$ and $Gen(P_R) = \overline{Gen}(P_R)$ and so P_R is a quasi-progenerator by theorem 2.2.5 .
(4) $\Rightarrow$ (1) Clear .

part (b)
(1) $\Rightarrow$ (2) By part (a) , P_R is a quasi-progenerator i.e. we have $Mod\text{-}A \sim Gen(P_R)$. And now $Gen(P_R) = Mod\text{-}R$, thus , the equivalence is actually between $Mod\text{-}A$ and $Mod\text{-}R$. Hence P_R is a progenerator .
(2) $\Rightarrow$ (1) Clear . □

3.2 Torsion Theories and $*$-modules

Let P_R be a $*$-module , $A = End(P_R)$ and $K_A = Hom_R(P, Q_R)$ for a cogenerator Q_R of $Mod\text{-}R$. Then by corollary 3.1.6 ,

$$H_P = Hom_R(P, -) : Gen(P_R) \longrightarrow Im(H_P)$$

is an exact functor . In fact ,

$$T_P = (- \otimes_A P) : Cogen(K_A) \longrightarrow Im(T_P)$$

is also an exact functor . Before we prove this , we first give the following useful result :

Proposition 3.2.1 [CM , proposition 1.1] *Let ${}_AP_R$ be a bimodule , $T_P = (- \otimes_A P)$, $H_P = Hom_R(P, -)$ with associated natural morphisms σ and ρ . Let $\mathcal{G}_R \subseteq Mod\text{-}R$ be a full subcategory closed under taking factor modules (or submodules) . If ρ_M is an isomorphism for every $M \in \mathcal{G}_R$ then T_P preserves short exact sequences of modules in $H_P(\mathcal{G}_R)$.*

Proof :
Let $0 \longrightarrow L \xrightarrow{f} K \longrightarrow N \longrightarrow 0$ be an exact sequence such that L , K and $N \in H_P(\mathcal{G}_R)$. Applying T_P , we get the exact sequence

$$T_P(L) \xrightarrow{T_P(f)} T_P(K) \rightarrow T_P(N) \rightarrow 0$$

in $\mathcal{G}_R$ because $\rho_M : T_P H_P(M) \cong M$ for every $M \in \mathcal{G}_R$. We decompose $T_P(f) : T_P(L) \xrightarrow{g} Im(T_P(f)) \xrightarrow{h} T_P(K)$ such that $T_P(f) = hg$ with g epic and h monic . Then

$$0 \rightarrow Im(T_P(f)) \xrightarrow{h} T_P(K) \rightarrow T_P(N) \rightarrow 0$$

is exact in $\mathcal{G}_R$, here $Im(T_P(f)) \in \mathcal{G}_R$ because $\mathcal{G}_R$ is closed under factor modules (or submodules) . As H_P is left exact , we obtain the commutative diagram with exact rows in $H_P(\mathcal{G}_R)$:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "H_P(g)\circ\sigma_L"'] & K \arrow[r] \arrow[d, "\sigma_K"] & N \arrow[r] \arrow[d, "\sigma_N"] & 0 \\
0 \arrow[r] & H_P(Im(T_P(f))) \arrow[r] & H_P T_P(K) \arrow[r] & H_P T_P(N) &
\end{tikzcd}$$

where σ_L , σ_K and σ_N are isomorphisms because

$$H_P(\rho_M) \circ \sigma_{H_P(M)} = Id_{H_P(M)}$$

for every $M \in \mathcal{G}_R$ and from the diagram $H_P(g) \circ \sigma_L$ is an isomorphism , hence $H_P(g)$ is an isomorphism . Now consider the commutative diagram

$$\begin{tikzcd}
T_P(L) \arrow[r, "g"] & Im(T_P(f)) \\
T_P H_P T_P(L) \arrow[r, "T_P H_P(g)"] \arrow[u, "\rho_{T_P(L)}"] & T_P H_P(Im(T_P(f))) \arrow[u, "\rho_{Im(T_P(f))}"']
\end{tikzcd}$$

where $\rho_{T_P(L)}$, $\rho_{Im(T_P(f))}$, $T_P H_P(g)$ are isomorphisms and hence g is an isomorphism . But $T_P(f) = h \circ g$ and so $T_P(f)$ is monic i.e. T_P preserves short exact sequences of modules in $H_P(\mathcal{G}_R)$. $\square$

Now we have

Proposition 3.2.2 [CM , proposition 1.2] *Let P_R be a $*$-module , $A = End(P_R)$ and $K_A = Hom_R(P, Q_R)$ for a cogenerator Q_R of $Mod\text{-}R$. Then*

(1) *the functor $T_P = (- \otimes_A P)$ preserves short exact sequences of modules in $Cogen(K_A)$.*

(2) $Cogen(K_A) = \{L \in Mod\text{-}A \mid Tor_1^A(L_A, {}_AP) = 0\} = Ker(Tor_1^A(-, {}_AP))$.

(3) $(Ker(T_P), Cogen(K_A))$ *is a torsion theory in $Mod\text{-}A$, with torsion class $= Ker(T_P)$ and torsion-free class $= Cogen(K_A)$.*

Proof :
(1) Put $\mathcal{G}_R = Gen(P_R)$ and then $H_P(\mathcal{G}_R) = Cogen(K_A)$. Now apply proposition 3.2.1 .
(2) Let $L \in Mod\text{-}A$ and $0 \longrightarrow N \xrightarrow{i} A^{(X)} \xrightarrow{\pi} L \longrightarrow 0$ be exact in $Mod\text{-}A$ and we have $A^{(X)}$ and $N \in Cogen(K_A)$. Applying T_P we get the long exact sequence [R , 5.2.28] ,

$$0 = Tor_1^A(A^{(X)}, {}_AP) \xrightarrow{\alpha = Tor_1^A(\pi, {}_AP)} Tor_1^A(L, {}_AP) \xrightarrow{\delta} T_P(N)$$
$$\xrightarrow{T_P(i)} T_P(A^{(X)}) \xrightarrow{T_P(\pi)} T_P(L) \longrightarrow 0$$

and note here that $A^{(X)}$ is a flat right A-module , so $0 = Tor_1^A(A^{(X)}, {}_AP)$.

Now if $L \in Cogen(K_A)$ then by (1) , $T_P(i)$ is monic and so $\delta = 0$. Hence $0 = Im(\alpha) = Ker(\delta) = Tor_1^A(L, {}_AP)$.

Conversely if $Tor_1^A(L, {}_AP) = 0$ then $T_P(i)$ is monic and

$$0 \longrightarrow T_P(N) \xrightarrow{T_P(i)} T_P(A^{(X)}) \xrightarrow{T_P(\pi)} T_P(L) \longrightarrow 0$$

is exact in $Gen(P_R)$. So by corollary 3.1.6 ,

$$0 \longrightarrow H_PT_P(N) \xrightarrow{H_PT_P(i)} H_PT_P(A^{(X)}) \xrightarrow{H_PT_P(\pi)} H_PT_P(L) \longrightarrow 0$$

is exact and then we have the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "i"] \arrow[d, "\sigma_N"', "\cong"] & A^{(X)} \arrow[r, "\pi"] \arrow[d, "\sigma_{A^{(X)}}"', "\cong"] & L \arrow[r] \arrow[d, "\sigma_L"] & 0 \\
0 \arrow[r] & H_PT_P(N) \arrow[r, "H_PT_P(i)"] & H_PT_P(A^{(X)}) \arrow[r, "H_PT_P(\pi)"] & H_PT_P(L) \arrow[r] & 0
\end{tikzcd}$$

where $\sigma_{A^{(X)}}$ and σ_N are isomorphisms because $A^{(X)}$ and $N \in Cogen(K_A)$. Then

$$L \cong H_P T_P(L) \in Im(H_P) = Cogen(K_A) \quad .$$

(3) By (2) , it follows that $Cogen(K_A) = Ker(Tor_1^A(-, {}_AP))$ is closed under extensions and hence it is a torsion-free class . Now we prove that its associated torsion class

$$^{\perp}(Cogen(K_A)) = Ker(T_P) = \{N_A \mid N \otimes_A P = 0\} \quad .$$

Let $N \in Mod\text{-}A$,

$$Hom_A(N, K) = Hom_A(N, Hom_R(P, Q)) \cong Hom_R(N \otimes_A P, Q) \quad .$$

So $N \in {}^{\perp}(Cogen(K_A))$ iff (as $Hom_A(N, -)$ preserves monomorphisms) , $Hom_A(N, K) = 0$ iff $Hom_R(N \otimes_A P, Q) = 0$ iff (as Q_R is a cogenerator) $N \otimes_A P = 0$ i.e. $N \in Ker(T_P)$. □

It should be noted that if $Gen(P_R)$ is closed under taking extensions then it is a torsion class and the corresponding torsion-free class is

$$Gen(P_R)^{\perp} = Ker(H_P) = \{L_R \mid Hom_R(P, L) = 0\}$$

since $L_R \in Gen(P_R)^{\perp}$ iff $Hom_R(P, L) = 0$.

If P_R is a $*$-module , then it is natural to ask when $Gen(P_R)$ is closed under taking extensions . In fact we have the following proposition due to Colpi .

Proposition 3.2.3 [C1 , proposition 4.4] *Let P_R be a $*$-module . Then the following conditions are equivalent :*

(1) $Ext_R^1(P, M) = 0$ *for any* $M \in Gen(P_R)$.

(2) $Gen(P_R)$ *is closed under extensions* .

Proof :
See [C1 , proposition 4.4] . □

3.3 The Structure of $*$-modules

Here we prove that every $*$-module is small [CM] and actually finitely generated [T2] .

Definition 3.3.1 *A module $P_R \in Mod\text{-}R$ is small if for any family $\{M_i\}_{i\in I}$ of right R-modules the canonical morphism*

$$\bigoplus_{i\in I} Hom_R(P, M_i) \xrightarrow{\eta} Hom_R(P, \bigoplus_{i\in I} M_i)$$

is an isomorphism where

$$\eta((g_i)_{i\in I})(p) = (g_i(p))_{i\in I}$$

for every $(g_i)_{i\in I} \in \bigoplus_{i\in I} Hom_R(P, M_i)$ and $p \in P$.

If we let $\pi_i : \bigoplus_{i\in I} M_i \rightarrow M_i$ be the i-th canonical projection for $i \in I$ then P_R is small if and only if for any $f \in Hom_R(P, \bigoplus_{i\in I} M_i)$ it turns out that $\pi_i \circ f \neq 0$ for only finitely many $i \in I$. Clearly every finitely generated module is small .

Lemma 3.3.2 [MO , proposition 3.4] *Let P_R be a $*$-module and $A = End(P_R)$. Then $H_P = Hom_R(P, -)$ commutes with all direct sums in $Gen(P_R)$.*

Proof :
Let $L_i \in Gen(P_R)$ for $i \in I$ then $\bigoplus_{i\in I} L_i \in Gen(P_R)$. Now there are $K_i \in Cogen(K_A)$ for $i \in I$ such that $L_i \cong T_P(K_i)$ and hence $H_P(L_i) \cong H_P T_P(K_i) \cong K_i$. We have

$$\begin{aligned} H_P(\textstyle\bigoplus_{i\in I} L_i) &\cong H_P(\textstyle\bigoplus_{i\in I} T_P(K_i)) \\ &\cong H_P(T_P(\textstyle\bigoplus_{i\in I} K_i)) \\ &\cong \textstyle\bigoplus_{i\in I} K_i \\ &\cong \textstyle\bigoplus_{i\in I} H_P(L_i) \end{aligned}$$

□

Proposition 3.3.3 [CM , proposition 1.8] *Every $*$-module P_R is small .*

Proof :
For any $f \in Hom_R(P, \bigoplus_{i\in I} M_i)$ and let $\pi_i : \bigoplus_{i\in I} M_i \longrightarrow M_i$ be the canonical projection . Note that

$$Im(f) \subseteq \bigoplus_{i\in I} Im(\pi_i \circ f) \subseteq \bigoplus_{i\in I} t_P(M_i)$$

and clearly both $\bigoplus_{i\in I} t_P(M_i)$ and $t_P(M_i) \in Gen(P_R)$. Then

$$\begin{aligned} Hom_R(P, \textstyle\bigoplus_{i\in I} M_i) &\cong Hom_R(P, \textstyle\bigoplus_{i\in I} t_P(M_i)) \\ &\cong \textstyle\bigoplus_{i\in I} Hom_R(P, t_P(M_i)) \quad \text{(by lemma 3.3.2)} \\ &\cong \textstyle\bigoplus_{i\in I} Hom_R(P, M_i) \end{aligned}$$

□

In [T2 , 1994] , Trlifaj proved that every $*$-module is actually finitely generated . This is one of the most important results about the structure of $*$-modules . He proves this fact by two lemmas .

Lemma 3.3.4 [T2 , lemma 2] *Let P_R be a $*$-module , $A = End(P_R)$ and $K_A = Hom_R(P, Q)$ for a cogenerator Q_R in Mod-R . Then $H_P = Hom_R(P, -)$ commutes with all direct limits in $Gen(P_R)$.*

Proof :
Let $(M_i, f_{i,j})_I$ be a directed system of modules such that $M_i \in Gen(P_R)$ for $i \in I$. Then by proposition 3.2.2(2) ,

$$H_P(M_i) \in Cogen(K_A) = \{L_A \mid Tor_1^A(L, P) = 0\}$$

which is closed under direct limits [R , remark 5.2.53] and so

$$\varinjlim H_P(M_i) \in Cogen(K_A) \ .$$

Since $Gen(P_R)$ is closed under direct sums and quotients ,

$$\varinjlim M_i \in Gen(P_R)$$

and then

$$\begin{aligned} H_P(\varinjlim M_i) &\cong H_P(\varinjlim T_P H_P(M_i)) \\ &\cong H_P T_P(\varinjlim H_P(M_i)) \overset{\rho}{\cong} \varinjlim H_P(M_i) \ . \ \square \end{aligned}$$

Lemma 3.3.5 [T2 , lemma 3] *Let P_R be a module such that $Hom_R(P,-)$ commutes with all direct limits in $Gen(P_R)$. Then P_R is finitely generated .*

Proof :
Let $P = \langle\ g_i \mid i \in I\ \rangle_R = \sum_{i\in I} g_iR$ for some set I . Since $g_iR \leq P$ for every $i \in I$, we have the external direct sum $\bigoplus_{i\in I} g_iR \leq P^{(I)}$ and the excat sequence

$$0 \to K \to \bigoplus_{i\in I} g_iR \xrightarrow{\theta} \sum_{i\in I} g_iR = P \to 0$$

where $K = Ker(\theta)$. Since $K \leq \bigoplus_{i\in I} g_iR \leq P^{(I)}$ we have

$$P \overset{\varphi}{\cong} (\bigoplus_{i\in I} g_iR)/K \xrightarrow{\mu\,=\,inc} P^{(I)}/K$$

and a monomorphism $\phi \overset{def}{=} \mu \circ \varphi \in Hom_R(P, P^{(I)}/K)$. Let $\lambda_i : P \to P^{(I)}$ be the i-th canonical injection and so $\phi(g_i) = \lambda_i(g_i) + K$ for every $i \in I$.

Now we consider the directed system , $(K_\alpha, i_{\alpha,\beta})$, of all finitely generated submodules of K and here $i_{\alpha,\beta} : K_\alpha \to K_\beta$ are inclusions for $K_\alpha \leq K_\beta$. Then

$$K = \varinjlim K_\alpha$$

and for $K_\alpha \leq K_\beta \leq K \leq P^{(I)}$ we get the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_\alpha & \longrightarrow & P^{(I)} & \longrightarrow & P^{(I)}/K_\alpha & \longrightarrow & 0 \\ & & {\scriptstyle i_{\alpha,\beta}}\big\downarrow & & =\big\downarrow & & \big\downarrow{\scriptstyle \pi_{\alpha,\beta}} & & \\ 0 & \longrightarrow & K_\beta & \longrightarrow & P^{(I)} & \longrightarrow & P^{(I)}/K_\beta & \longrightarrow & 0 \end{array}$$

where $\pi_{\alpha,\beta}(x + K_\alpha) \overset{def}{=} x + K_\beta$ for every $x + K_\alpha \in P^{(I)}/K_\alpha$.

By [W , 24.6] , for the directed system $(P^{(I)}/K_\alpha, \pi_{\alpha,\beta})$, we get the exact sequence

$$0 \to \varinjlim K_\alpha \to \varinjlim P^{(I)} \to \varinjlim P^{(I)}/K_\alpha \to 0$$

and that is

$$0 \to K \to P^{(I)} \to \varinjlim P^{(I)}/K_\alpha \to 0$$

. Hence we have

$$P^{(I)}/K = \varinjlim(P^{(I)}/K_\alpha)$$

i.e. the direct limit of the directed system $(P^{(I)}/K_\alpha, \pi_{\alpha,\beta})$ is $(P^{(I)}/K, \pi_\alpha)$ where $\pi_\beta : P^{(I)}/K_\beta \longrightarrow P^{(I)}/K$ is defined by $\pi_\beta(x + K_\beta) = x + K$ for every $x + K_\beta \in P^{(I)}/K_\beta$.

Since $P^{(I)}/K_\alpha \in Gen(P_R)$ for every α and $H_P = Hom_R(P, -)$ preserves all direct limits in $Gen(P_R)$, so the direct limit of the directed system $(Hom_R(P, P^{(I)}/K_\alpha), Hom_R(P, \pi_{\alpha,\beta}))$ is $(Hom_R(P, P^{(I)}/K), Hom_R(P, \pi_\alpha))$ i.e.

$$\varinjlim Hom_R(P, P^{(I)}/K_\alpha) \cong Hom_R(P, \varinjlim P^{(I)}/K_\alpha) = Hom_R(P, P^{(I)}/K)$$

Now for $\phi = \mu \circ \varphi \in Hom_R(P, P^{(I)}/K)$, by [W , 24.3(2)] , there exists $\phi_\beta \in Hom_R(P, P^{(I)}/K_\beta)$ such that $\phi = \pi_\beta \circ \phi_\beta$.

As $P^{(I)} \geq K_\beta$ which is finitely generated so $K_\beta \leq P^{(F)}$ for some finite set $F \subseteq I$ and we get

$$P^{(I)}/K_\beta = (P^{(F)}/K_\beta) \bigoplus P^{(I\setminus F)}$$

. Thus ,

$$\phi_\beta \in Hom_R(P, P^{(I)}/K_\beta) = Hom_R(P, (P^{(F)}/K_\beta) \bigoplus P^{(I\setminus F)})$$

and since P is small , there is a finite set H such that $F \subseteq H \subseteq I$ and

$$Im(\phi_\beta) \subseteq (P^{(F)}/K_\beta) \bigoplus P^{(H\setminus F)} = P^{(H)}/K_\beta$$

$$Im(\phi) = \pi_\beta(Im(\phi_\beta)) \subseteq \pi_\beta(P^{(H)}/K_\beta)$$

. Clearly $\pi_\beta(P^{(H)}/K_\beta) \subseteq (P^{(H)} + K)/K$ and so $Im(\phi) \subseteq (P^{(H)} + K)/K$ and $\phi(g_i) = p_i + K$ for some $p_i \in P^{(H)}$. But $\phi(g_i) = \lambda_i(g_i) + K$ where $\lambda_i(g_i) \in \bigoplus_{i\in I} g_iR \subseteq P^{(I)}$. Then since $\lambda_i(g_i) - p_i \in K \subseteq \bigoplus_{i\in I} g_iR$ we get $p_i \in \bigoplus_{i\in I} g_iR$ and hence

$$p_i \in P^{(H)} \cap \bigoplus_{i\in I}(g_iR) = \bigoplus_{i\in H}(g_iR)$$

. Finally we obtain that

$$Im(\phi) = (\bigoplus_{i\in H} g_iR + K)/K = \bigoplus_{i\in H} \phi(g_i)R$$

which is finitely generated since H is a finite set . So $P \cong Im(\phi)$ is finitely generated . $\square$

It is worthy to point that for an R-module P , $Hom_R(P, -)$ commutes with all direct limits iff P is finitely presented [W , 25.4] .

Immediately we obtain the important

Theorem 3.3.6 [T2 , theorem 1 , 1994] *Every $*$-module over an arbitrary ring is finitely generated .*

Finally we have the following characterizations of $*$-modules .

Corollary 3.3.7 *Let P_R be a module . Then the following conditions are equivalent :*

(1) *P_R is a $*$-module .*

(2) *P_R is finitely generated , w-$\sum$-quasi-projective and $Gen(P_R) = Pres(P_R)$.*

(3) *P_R is finitely generated and for exact sequences $0 \longrightarrow M_R \xrightarrow{i} P_R^{(\Lambda)}$, where Λ is a cardinal , then $M \in Gen(P_R)$ if and only if*

$$Ext_R^1(P, i) : Ext_R^1(P, M) \longrightarrow Ext_R^1(P, P_R^{(\Lambda)})$$

is monic .

(4) *P_R is finitely generated , $Gen(P_R) = Pres(P_R)$ and for every exact sequence $0 \longrightarrow M_R \xrightarrow{i} P_R^{(\Lambda)}$, where Λ is a cardinal , then $M \in Gen(P_R)$ implies that $Ext_R^1(P, i) : Ext_R^1(P, M) \longrightarrow Ext_R^1(P, P_R^{(\Lambda)})$ is monic .*

(5) *P_R is finitely generated and for any exact sequence in Mod-R*

$$(ex) : \quad 0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

with $M \in Gen(P_R)$, then $H_P(ex)$ is exact if and only if $L \in Gen(P_R)$.

3.4 Characterizations of Tilting Modules

In [MO] , Menini and Orsatti have provided important examples , namely non-projective tilting modules , of $*$-modules that are not quasi-progenerators . In this section , we study the tilting modules and give some characterizations of them and the results are mainly due to Colpi [C2 , 1993] . The notion of tilting modules originally come from the representation theory of finite dimensional algebras .

Let P be a module , $Add(P)$ ($add(P)$) denotes the class of all summands of (finite) direct sums of copies of P . For a module M , $I(M)$ denotes the injective envelope of M .

Definition 3.4.1 *$P_R \in Mod\text{-}R$ is called a tilting module if*

(1) P_R is finitely presented and $proj.dim(P_R) \leq 1$.

(2) $Ext^1_R(P, P) = 0$.

(3) there is an exact sequence $0 \to R \to P' \to P'' \to 0$, where P' and P'' are in $Add(P_R)$.

We begin with the

Lemma 3.4.2 [T1 , lemma 1.1] *If P_R is a countably generated small module then P_R is finitely generated .*

Proof :
Suppose P_R is NOT finitely generated . Then there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq M$ such that for every n , $x_n \in M \setminus \langle\, x_1, ..., x_{n-1} \,\rangle_R$ and $\langle\, x_n \mid n \in \mathbb{N} \,\rangle_R = P$ because P is countably generated .

Let Q_R be an *injective* cogenerator of $Mod\text{-}R$ and because

$$0 \to P/\langle\, x_1, ..., x_{n-1} \,\rangle_R \ \to Q^Y$$

is exact for some set Y , there exists a morphism

$$\psi_n : P/\langle\, x_1, ..., x_{n-1} \,\rangle_R \ \to Q$$

such that $\psi_n(x_n + \langle\, x_1, ..., x_{n-1} \,\rangle_R) \neq 0$. Define

$$\phi_n = \psi_n \circ \pi : P \xrightarrow{\pi} P/\langle\, x_1, ..., x_{n-1} \,\rangle_R \xrightarrow{\psi_n} Q$$

where $\pi =$ the natural epimorphism . So $\phi_n(x_n) \neq 0$ and $\phi_n(x_i) = 0$ for all $1 \leq i \leq n-1$. Now we have

$$\phi \overset{def}{=} \prod_n \phi_n : P \longrightarrow Q^{\mathbb{N}}$$

and since $P = \langle x_1, ..., x_n, ... \rangle_R$ and by construction $\phi(P) \subseteq Q^{(\mathbb{N})} \subseteq Q^{\mathbb{N}}$ so we have $\phi \in Hom_R(P, Q^{(\mathbb{N})})$ but $Im(\phi) \nsubseteq Q^{(F)}$ for any finite subset $F \subseteq \mathbb{N}$ because $\phi_n(x_n) \neq 0$ for all n . Thus , P_R is NOT small . $\square$

Proposition 3.4.3 [CM , proposition 1.7] *Let P_R be a module . If $P^{\perp} \overset{def}{=} \{M_R \mid Ext^1_R(P, M) = 0\}$ is closed under direct sums , $proj.dim(P_R) \leq 1$ and P_R is finitely generated then P_R is finitely presented .*

Proof :
Now P_R is finitely generated and $proj.dim(P_R) \leq 1$ we have an exact sequence

$$(ex) \;:\; 0 \longrightarrow L_R \xrightarrow{i} R^{(n)} \longrightarrow P \longrightarrow 0$$

where $i =$ inclusion , $n \in \mathbb{N}$ and L_R is projective i.e. L_R is a direct summand of a direct sum of copies of P and so we have a splitting monomorphism $0 \longrightarrow L \xrightarrow{j} R^{(X)}$ for some set X . We need to show that L is finitely generated .

Let $\lambda = inclusion : R^{(X)} \longrightarrow (I(R))^{(X)}$, where $I(R)$ is the injective envelope of R_R , and $f = \lambda \circ j : L \longrightarrow (I(R))^{(X)}$. As $I(R) \in P^{\perp}$ which is closed under direct sums so $Ext^1_R(P, (I(R))^{(X)}) = 0$. Apply $Hom_R(-, (I(R))^{(X)})$ to (ex) , we have the exact sequence

$$0 \longrightarrow Hom_R(P, (I(R))^{(X)}) \longrightarrow Hom_R(R^{(n)}, (I(R))^{(X)})$$

$$\xrightarrow{i^*} Hom_R(L, (I(R))^{(X)}) \longrightarrow Ext^1_R(P, (I(R))^{(X)}) = 0$$

, so for $f \in Hom_R(L, (I(R))^{(X)})$ there is a $\overline{f} \in Hom_R(R^{(n)}, (I(R))^{(X)})$ such that $\overline{f} \circ i = f$.

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "i"'] & (I(R))^{(X)} \\ R^{(n)} \arrow[r, "="] & R^{(n)} \arrow[u, "\exists \overline{f}"'] \end{tikzcd}$$

Consider

$$R^{(n)} \xrightarrow{\overline{f}} (I(R))^{(X)} \xrightarrow{\pi_x} I(R)$$

and clearly $R^{(n)}$ is finitely generated so , as $Im(\overline{f})$ is finitely generated , the set

$$F \overset{def}{=} \{x \in X \mid \pi_x \circ \overline{f} \neq 0\}$$

is finite . Let $q_x : R^{(X)} \longrightarrow R$ be the canonical projection . Clearly we have the commutative diagram

$$\begin{array}{ccccc} L & \xrightarrow{j} & R^{(X)} & \xrightarrow{\lambda} & (I(R))^{(X)} \\ & & {\scriptstyle q_x}\downarrow & & \downarrow{\scriptstyle \pi_x} \\ & & R & \xrightarrow{inc} & I(R) \end{array}$$

. Now for any $x \in X \setminus F$ i.e. $\pi_x \circ \overline{f} = 0$ and hence $\pi_x \circ f = 0$ because $\overline{f} \circ i = f$. As $f = \lambda \circ j$, we have

$$0 = \pi_x \circ f = \pi_x \circ \lambda \circ j = inc \circ q_x \circ j$$

and this means that $Im(j) \subseteq R^{(F)}$, therefore L is isomorphic to a direct summand of $R^{(F)}$ and since F is a finite set , L is finitely generated . □

Proposition 3.4.4 [T1 , lemma 1.2] *Let P_R be a module . Then the following conditions are equivalent :*

(1) *P_R is finitely presented and $proj.dim(P_R) \leq 1$.*

(2) *P_R is small and*

$$P^{\perp} \overset{def}{=} Ker(Ext^1_R(P,-)) = \{M_R \mid Ext^1_R(P,M) = 0\}$$

is closed under direct sums and factors .

Proof :
(1) ⇒ (2) P is finitely generated implies that P is small .
Let $M_R \in P^{\perp}$ i.e. $Ext^1_R(P,M) = 0$ and

$$0 \to K \to M \to M/K \to 0$$

be an exact sequence . Applying $Hom_R(P, -)$ we get the long exact sequence

$$0 \longrightarrow Hom_R(P, K) \longrightarrow Hom_R(P, M) \longrightarrow Hom_R(P, M/K)$$
$$\longrightarrow Ext^1_R(P, K) \longrightarrow Ext^1_R(P, M) = 0 \longrightarrow Ext^1_R(P, M/K)$$
$$\longrightarrow Ext^2_R(P, K) = 0$$

. Note here that $proj.dim(P_R) \leq 1$ if and only if $Ext^2_R(P, -) = 0$ and now we have $Ext^1_R(P, M/K) = 0$ i.e. $P^\perp$ is closed under factors .

Since P is finitely presented and $proj.dim(P_R) \leq 1$, we have an exact sequence

$$(se): \quad 0 \longrightarrow Y_R \xrightarrow{j} X_R \longrightarrow P_R \longrightarrow 0$$

where Y_R is finitely generated and X_R is projective . Clearly $P^\perp$ is closed under finite direct sums . Apply $Hom_R(-, \bigoplus_{i \in I} M_i)$ to (se) , where $M_i \in P^\perp$ for all $i \in I$, we get the long exact sequence

$$0 \longrightarrow Hom_R(P, \bigoplus_{i \in I} M_i) \longrightarrow Hom_R(X, \bigoplus_{i \in I} M_i) \xrightarrow{j^*} Hom_R(Y, \bigoplus_{i \in I} M_i)$$

$$\longrightarrow Ext^1_R(P, \bigoplus_{i \in I} M_i) \longrightarrow Ext^1_R(X, \bigoplus_{i \in I} M_i) = 0$$

. Now we claim that $j^* = Hom_R(j, \bigoplus_{i \in I} M_i)$ is epic and hence $Ext^1_R(P, \bigoplus_{i \in I} M_i) = 0$.

To see this , let $g \in Hom_R(Y, \bigoplus_{i \in I} M_i)$, since Y is finitely generated , $Im(g) \subseteq \bigoplus_{i \in F} M_i$ for some finite set $F \subseteq I$. But now the sequence

$$0 \longrightarrow Hom_R(P, \bigoplus_{i \in F} M_i) \longrightarrow Hom_R(X, \bigoplus_{i \in F} M_i)$$

$$\xrightarrow{Hom_R(j, \bigoplus_{i \in F} M_i)} Hom_R(Y, \bigoplus_{i \in F} M_i) \longrightarrow Ext^1_R(P, \bigoplus_{i \in F} M_i) = 0$$

is exact and here $Ext^1_R(P, \bigoplus_{i \in F} M_i) = 0$ because $P^\perp$ is closed under finite direct sums . Then there exists a $h \in Hom_R(X, \bigoplus_{i \in F} M_i)$ such that $h \circ j = g$

$$\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "j"'] & \bigoplus_{i \in F} M_i \\
X \arrow[r, "="] & X \arrow[u, "\exists\, h"']
\end{tikzcd}$$

, here we can consider h as a morphism $X \to \bigoplus_{i\in I} M_i$ and so $j^* = Hom_R(j, \bigoplus_{i\in I} M_i)$ is epic .

Therefore $Ext^1_R(P, \bigoplus_{i\in I} M_i) = 0$ i.e. $P^\perp$ is closed under direct sums .

(2) $\Rightarrow$ (1)

(i) We first show that $proj.dim(P_R) \leq 1$.

Consider the exact sequence

$$(ex1): \quad 0 \to L_R \to K_R \to P \to 0$$

with K_R projective , we claim that L_R is also projective and hence $proj.dim(P_R) \leq 1$ [R , 5.1.8(ii)] .

Let $N \in Mod\text{-}R$, apply $Hom_R(-, N)$ to $(ex1)$ and note that K_R is projective , we get the long exact sequence

$$0 \to Hom_R(P,N) \to Hom_R(K_R,N) \to Hom_R(L,N)$$
$$\to Ext^1_R(P,N) \to Ext^1_R(K_R,N) = 0 \to Ext^1_R(L,N)$$
$$\to Ext^2_R(P,N) \to Ext^2_R(K_R,N) = 0$$

and so $Ext^1_R(L,N) \cong Ext^2_R(P,N)$ as abelian groups .

Now we consider the exact sequence

$$(ex2): \quad 0 \to N \to I(N) \to N/I(N) \to 0$$

where $I(N)$ is the injective hull of N . Applying $Hom_R(P,-)$ to $(ex2)$ and since clearly $I(N)$ is injective , we have a long exact sequence

$$0 \to Hom_R(P,N) \to Hom_R(P,I(N)) \to Hom_R(P,I(N)/N)$$
$$\to Ext^1_R(P,N) \to Ext^1_R(P,I(N)) = 0 \to Ext^1_R(P,I(N)/N)$$
$$\to Ext^2_R(P,N) \to Ext^2_R(P,I(N)) = 0$$

and so $Ext^1_R(P,I(N)/N) \cong Ext^2_R(P,N)$ as abelian groups .

Since $I(N) \in P^\perp$ and by assumption , $P^\perp$ is closed under quotients , $I(N)/N \in P^\perp$ and hence

$$Ext^1_R(L,N) \cong Ext^2_R(P,N) \cong Ext^1_R(P,I(N)/N) = 0$$

. Therefore $Ext^1_R(L,-)=0$ and so L is projective .
(ii) We need to show that P is finitely presented .
Consider the exact sequence

$$(ex3): \quad 0 \longrightarrow L \xrightarrow{inc} R^{(I)} \xrightarrow{\mu} P \longrightarrow 0$$

, where $inc =$ inclusion and by (i) , L_R is projective . By [AF , corollary 26.2] , we can write $L = \bigoplus_{j\in J} L_j$, where for each j , L_j is countably generated and projective . Let $D = \bigoplus_{j\in J} I(L_j)$ and $i : L \longrightarrow D$ be the canonical inclusion map . Since $P^{\perp}$ is closed under direct sums by assumption , $Ext^1_R(P,D)=0$. Apply $Hom_R(-,D)$ to $(ex3)$, we have the exact sequence

$$0 \longrightarrow Hom_R(P,D) \longrightarrow Hom_R(R^{(I)},D) \xrightarrow{inc^*} Hom_R(L,D) \longrightarrow Ext^1_R(P,D)=0$$

, so there is a $g \in Hom_R(R^{(I)},D)$ such that $g \circ inc = i$.

$$\begin{array}{ccc} L & \xrightarrow{i} & D \\ {\scriptstyle inc}\downarrow & & \uparrow{\scriptstyle \exists\, g} \\ R^{(I)} & \xrightarrow{=} & R^{(I)} \end{array}$$

Consider the situation :

$$R^{(I)} \xrightarrow{g} D = \bigoplus_{j\in J} I(L_j) \xrightarrow{\pi_j} I(L_j) \xrightarrow{q_j} I(L_j)/L_j$$

where $\pi_j =$ *canonical projection* and $q_j =$ *natural epimorphism* and

$$g_j \stackrel{def}{=} q_j \circ \pi_j \circ g : R^{(I)} \longrightarrow I(L_j)/L_j$$

. Since for every $x \in R^{(I)}$, there are only finitely many $j \in J$ such that $g_j(x) = q_j(\pi_j(g(x))) \neq 0$ and so we can define

$$h : R^{(I)} \longrightarrow \bigoplus_{j\in J} I(L_j)/L_j$$

by $h(x) = [q_j(\pi_j(g(x)))]_{j\in J} \in \bigoplus_{j\in J} I(L_j)/L_j$ for $x \in R^{(I)}$.

Since $Ker(\mu) = L \subseteq Ker(h)$ we have a $\overline{h} \in Hom_R(P, \bigoplus_{j\in J} I(L_j)/L_j)$ such that $\overline{h} \circ \mu = h$.

$$\begin{tikzcd} R^{(I)} \arrow[r, "h"] \arrow[d, "epic\ \mu"'] & \bigoplus_{j\in J} I(L_j)/L_j \\ P \arrow[r, "="] & P \arrow[u, "\exists\ \overline{h}"'] \end{tikzcd}$$

By assumption , P is small , there is a finite subset $F \subseteq J$ such that

$$Im(\overline{h}) \subseteq \bigoplus_{j\in F} I(L_j)/L_j$$

and then

$$Im(g) \subseteq (\bigoplus_{j\in F} I(L_j)) \bigoplus (\bigoplus_{j\in J\setminus F} L_j)$$

. Now consider the projection

$$\nu : D = \bigoplus_{j\in J} I(L_j) \rightarrow \bigoplus_{j\in J\setminus F} I(L_j)$$

. If $\overline{g} \overset{def}{=} \nu \circ g : R^{(I)} \rightarrow \bigoplus_{j\in J\setminus F} I(L_j)$ and $\overline{L} \overset{def}{=} \bigoplus_{j\in J\setminus F} L_j \leq L$ then

$$\overline{g} \in Hom_R(R^{(I)}, \bigoplus_{j\in J\setminus F} L_j) = Hom_R(R^{(I)}, \overline{L})$$

. Note that $\overline{g}|_{\overline{L}} = Id_{\overline{L}}$ and hence $R^{(I)} = Ker(\overline{g}) \bigoplus \overline{L}$. Let $B = Ker(\overline{g}) \cap L = \bigoplus_{j\in F} L_j$. Then

$$P \cong R^{(I)}/L = (Ker(\overline{g}) + L)/L \cong Ker(\overline{g})/B \quad .$$

Since $Ker(\overline{g})$ is projective , by [AF , 26.2] , it is a direct sum of countably generated projective modules , so $Ker(\overline{g}) = \bigoplus_{j\in K} H_j$ for some set $K \supseteq J$ and here every H_j is countably generated . For $j \in K \setminus J$, we set $L_j = 0$ and then

$$\begin{aligned} P \cong R^{(I)}/L = (Ker(\overline{g}) + L)/L &\cong (\textstyle\bigoplus_{j\in K} H_j + \bigoplus_{j\in J} L_j)/(\bigoplus_{j\in J} L_j) \\ &\cong (\textstyle\bigoplus_{j\in K}(H_j + L_j))/(\bigoplus_{j\in K} L_j) \\ &\cong \textstyle\bigoplus_{j\in K}[(H_j + L_j)/L_j] \end{aligned}$$

$$i.e. \quad P \overset{\xi}{\cong} \bigoplus_{j \in K} [(H_j + L_j)/L_j]$$

for some ξ . But now $\xi \in Hom_R(P, \bigoplus_{j\in K}[(H_j + L_j)/L_j])$ and P is small , so there is a finite set $C \subseteq K$ such that $Im(\xi) \subseteq \bigoplus_{j\in C}[(H_j + L_j)/L_j]$ and this implies that

$$P \overset{\xi}{\cong} \bigoplus_{j \in C} [(H_j + L_j)/L_j]$$

which is countably generated . By lemma 3.4.2 and note that P is small , P is finitely generated .

Now P_R is finitely generated , $proj.dim(P_R) \leq 1$ and by proposition 3.4.3 , P_R is finitely presented . □

For a $*$-module P_R , there is a weak connection between the classes $Gen(P_R)$ and $Ker(Ext^1_R(P,-)) = \{M \in Mod\text{-}R \mid Ext^1_R(P,M) = 0\}$ shown by the following proposition .

Proposition 3.4.5 [C1 , proposition 4.5] *Let $P_R \in Mod\text{-}R$, then the following conditions are equivalent :*

(a) $\overline{Gen}(P_R) = Mod\text{-}R$.

(b) P_R generates any injective module .

If the previous conditions are satisfied , then the following statements are equivalent :

(1) *P_R is a $*$-module .*

(2) *P_R is selfsmall and*

$$Gen(P_R) = \{M \in Mod\text{-}R \mid Ext^1_R(P,M) = 0\} \overset{def}{=} P^{\perp}$$

Proof :
(a) $\Rightarrow$ (b) By hypothesis , $R \in \overline{Gen}(P_R) = Mod\text{-}R$, so we have an inclusion $0 \longrightarrow R \xrightarrow{i} M_R$ with $M \in Gen(P_R)$ and a commutative diagram

$$\begin{tikzcd} R \arrow[d, "\exists\, h"'] & R \arrow[l, "="'] \arrow[d, "i"] \\ P^{(X)} \arrow[r, "\pi"] & M \arrow[r] & 0 \end{tikzcd}$$

where h exists because R_R is projective . Also $\pi h = i$ monic implies that h is monic i.e. $0 \longrightarrow R \xrightarrow{h} P^{(X)}$ is exact .

Now let $E_R \in Mod\text{-}R$ be any injective module and we have an epimorphism

$$R^{(Y)} \xrightarrow{g} E \longrightarrow 0$$

. Consider the commutative diagram ,

$$\begin{tikzcd} 0 \arrow[r] & R^{(Y)} \arrow[r, "j"] \arrow[d, "\text{epic } g"'] & P^{(X\times Y)} \arrow[d, "\exists f"] \\ & E & E \arrow[l, "="'] \end{tikzcd}$$

where f exists because E_R is injective and $fj = g$ epic implies that f is epic . So $E \in Gen(P_R)$.

(b) $\Rightarrow$ (a) It is clear because every R-module is contained in its injective envelope .

Now we suppose that condition (b) holds .

(1) $\Rightarrow$ (2) Let $M \in Mod\text{-}R$ and $I(M)$ = injective envelope of M . Consider the exact sequence

$$0 \longrightarrow M \longrightarrow I(M) \xrightarrow{\pi} I(M)/M \longrightarrow 0$$

and applying $H_P = Hom_R(P, -)$, we get a long exact sequence

$$0 \longrightarrow H_P(M) \longrightarrow H_P(I(M)) \longrightarrow H_P(I(M)/M)$$

$$\xrightarrow{\delta} Ext^1_R(P, M) \longrightarrow Ext^1_R(P, I(M)) = 0$$

. By conditions (b) , $I(M) \in Gen(P_R)$ and now by proposition 3.1.7 ,

$M \in Gen(P_R)$

iff $H_P(\pi)$ is epic

iff $\delta = 0$

iff $\quad Ext^1_R(P, M) = 0$.

That is $Gen(P_R) = \{M \in Mod\text{-}R \mid Ext^1_R(P, M) = 0\}$.

(2) $\Rightarrow$ (1) We check the condition 3.1.7(2) .
For any exact sequence in $Mod\text{-}R$,

$$(ex): \quad 0 \longrightarrow L \longrightarrow M \xrightarrow{\pi} N \longrightarrow 0$$

where $M \in Gen(P_R) = P^{\perp}$, applying H_P , we get a long exact sequence

$$0 \longrightarrow H_P(L) \longrightarrow H_P(M) \xrightarrow{H_P(\pi)} H_P(N)$$

$$\xrightarrow{\delta} Ext^1_R(P, L) \longrightarrow Ext^1_R(P, M) = 0$$

hence δ is epic and now

$H_P(\pi)$ is epic

iff $\quad \delta = 0$

iff $\quad Ext^1_R(P, L) = 0$ i.e. $L \in P^{\perp} = Gen(P_R)$.

□

Definition 3.4.6 *P_R is said to be finendo if , for $A = End(P_R)$, ${}_AP$ is a finitely generated left A-module .*

Now we can give our main theorem which describes the relation between $*$-modules and tilting modules . Actually these are also characterizations of tilting modules and the results are due to Colpi [C2] .

Theorem 3.4.7 [C2 , theorem 3 , 1993] *Let $P_R \in Mod\text{-}R$ and $A = End(P_R)$. The following conditions are equivalent :*

(1) *P_R is a tilting module .*

(2) *P_R satisfies the conditions :*

(a) P_R is finitely presented and $proj.dim(P_R) \leq 1$,

(b) $Ext^1_R(P, P) = 0$ and

(c) *for every* $M \in Mod\text{-}R$, $Hom_R(P,M) = 0 = Ext^1_R(P,M)$ *implies that* $M = 0$.

(3) P_R *is finitely generated and*

$$Gen(P_R) = \{M_R \mid Ext^1_R(P,M) = 0\} = P^{\perp}$$

(4) P_R *is a faithful finendo* $*$*-module* .

(5) P_R *is a* $*$*-module and* $I(R_R) \in Gen(P_R)$.

Proof :
(1) $\Rightarrow$ (2) We need only to check the condition (c) . Let $M \in Mod\text{-}R$ and $Hom_R(P,M) = 0 = Ext^1_R(P,M)$, by assumption , we have an exact sequence

$$0 \longrightarrow R \longrightarrow P' \longrightarrow P'' \longrightarrow 0$$

, where P' and P'' are direct summands of a direct sum of copies of P_R and so $Hom_R(P',M) = 0$ and $Ext^1_R(P'',M) = 0$. Applying $Hom_R(-,M)$, we get a long exact sequence

$$0 \longrightarrow Hom_R(P'',M) \longrightarrow Hom_R(P',M) = 0 \longrightarrow Hom_R(R,M) \cong M$$

$$\xrightarrow{\delta} Ext^1_R(P'',M) = 0$$

and hence $M = 0$.

(2) $\Rightarrow$ (3) Clearly P_R is finitely generated .
Since $P_R \in P^{\perp}$ as $Ext^1_R(P,P) = 0$ and $P^{\perp}$ is closed under direct sums and factors by proposition 3.4.4 and condition (a) , so $Gen(P_R) \subseteq P^{\perp}$.
Now let $0 \neq M \in Mod\text{-}R$ such that $Ext^1_R(P,M) = 0$ and recall that $t_P(M) = \sum\{Imf \mid f \in Hom_R(P,M)\}$. So $t_P(M) \neq 0$ because if $t_P(M) = 0$ then $Hom_R(P,M) = 0 = Ext^1_R(P,M)$ and , by condition (c) , $M = 0$. Consider the exact sequence

$$0 \longrightarrow t_P(M) \longrightarrow M \longrightarrow M/t_P(M) \longrightarrow 0$$

and applying $Hom_R(P,-)$ we get the exact sequence

$$Ext^1_R(P,M) = 0 \longrightarrow Ext^1_R(P,M/t_P(M)) \longrightarrow Ext^2_R(P,t_P(M)) = 0$$

where $Ext^2_R(P, t_P(M)) = 0$ because $proj.dim(P_R) \leq 1$ by assumption . So $Ext^1_R(P, M/t_P(M)) = 0$. But $Hom_R(P, M/t_P(M)) = 0$ in general and hence by assumption (condition (c)) , $M/t_P(M)) = 0$ i.e. $M = t_P(M) \in Gen(P_R)$.

(3) $\Rightarrow$ (4) We check that P_R satisfies the condition (5) of corollary 3.3.7 and so P_R is a $*$-module .

Let (ex) : $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence in Mod-R with $M \in Gen(P_R) = P^{\perp}$. Applying $Hom_R(P, -)$ we get a long exact sequence

$$0 \longrightarrow Hom_R(P, L) \longrightarrow Hom_R(P, M) \longrightarrow Hom_R(P, N)$$

$$\xrightarrow{\delta} Ext^1_R(P, L) \longrightarrow Ext^1_R(P, M) = 0$$

. If $H_P(ex)$ is exact then δ = zero map and hence $Ext^1_R(P, L) = 0$ i.e. $L \in P^{\perp} = Gen(P_R)$.

Conversely if $L \in Gen(P_R) = P^{\perp}$ then clearly $H_P(ex)$ is exact .

As $I(R_R) \in P^{\perp} = Gen(P_R)$, we have an epimorphism $P_R^{(X)} \xrightarrow{h} I(R) \longrightarrow 0$ and the commutative diagram

$$\begin{tikzcd} R \arrow[d, "\exists\, \overline{h}"'] & R \arrow[l, "="'] \arrow[d, "inc"] & \\ P_R^{(X)} \arrow[r, "h"] & I(R) \arrow[r] & 0 \end{tikzcd}$$

because R is projective , there is a $\overline{h} : R \longrightarrow P_R^{(X)}$ such that $h \circ \overline{h} = inc$ and so $\overline{h}$ is monic . It follows that P_R is a faithful right R-module because if for some $r \in R$, $Pr = 0$ then $P_R^{(X)} r = 0$. Hence $Rr = 0$ and so $r = 0$.

Note that for any set Λ , $Ext^1_R(P, P^{\Lambda}) \cong Ext^1_R(P, P)^{\Lambda} = 0$ so $P_R^{\Lambda} \in Gen(P_R) = P^{\perp}$.

Now for the module P^P there is an epimorphism

$$\varphi : P^{(I)} \longrightarrow P^P \longrightarrow 0$$

for some set I . And we consider the element $(x_p)_p \in P^P$ such that $x_p = p$ for every $p \in P$. Then there exists a $(y_i)_i \in P^{(I)}$ such that

$$(x_p)_p = \varphi((y_i)_i) = \sum_{i \in F} \varphi_i(y_i)$$

here we write $\varphi = (\varphi_i)_{i\in I}$ for $\varphi_i \in Hom_R(P, P^P)$ and $F \stackrel{def}{=} \{i \in I \mid y_i \neq 0\}$ is a finite subset of I , note here that F is determined by the element $(x_p)_{p\in P}$.

Let $\pi_p : P^P \longrightarrow P$ be the canonical projection for every $p \in P$ and we have the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\varphi_i} & P^P \\ {\scriptstyle a_i^p=\pi_p\varphi_i}\downarrow & & \downarrow{\scriptstyle \pi_p} \\ P & \xrightarrow{=} & P \end{array}$$

where $a_i^p \stackrel{def}{=} \pi_p\varphi_i \in End(P_R) = A$.

Now for $p \in P$,

$$\begin{array}{rcll} p = \pi_p((x_p)_p) & = & \pi_p(\sum_{i\in F}\varphi_i(y_i)) & \\ & = & \sum_{i\in F}\pi_p\varphi_i(y_i) & \\ & = & \sum_{i\in F}a_i^p(y_i) & , \ a_i^p \in A \\ & \in & {}_A\langle\, y_i \mid i \in F\,\rangle & \end{array}$$

and so ${}_AP = {}_A\langle\, y_i \mid i \in F\,\rangle$.

(4) $\Rightarrow$ (5) Now by condition (4) we have the canonical ring embedding

$$0 \longrightarrow R \longrightarrow End({}_AP)$$

and an epimorphism

$$(ep): \ {}_AA^n \longrightarrow {}_AP \longrightarrow 0$$

for some $n \in \mathbb{N}$. Apply $Hom_A(-, {}_AP_R)$ to (ep) , we get an exact sequence

$$0 \longrightarrow Hom_A({}_AP, {}_AP_R) \longrightarrow Hom_A({}_AA^n, {}_AP_R) \cong P_R^n$$

. Since $R \leq End({}_AP) = Hom_A({}_AP, {}_AP_R)$ we obtain a monomorphism

$$0 \longrightarrow R \longrightarrow P_R^n$$

. Clearly R_R is a generator of Mod-R and so there is an epimorphism

$$R^{(X)} \xrightarrow{\phi} I(R_R) \longrightarrow 0$$

for some set X . Since we have an embedding $0 \longrightarrow R \longrightarrow P_R^n$ and then we get a monomorphism $0 \longrightarrow R^{(X)} \xrightarrow{i} P_R^{(Y)}$ for some set Y . Now we have a commutative diagram with exact rows and column

$$\begin{array}{ccccc} 0 & \longrightarrow & R^{(X)} & \xrightarrow{i} & P_R^{(Y)} \\ & & \phi\big\downarrow & & \big\downarrow \exists f \\ & & I(R_R) & \xrightarrow{=} & I(R_R) \\ & & \big\downarrow & & \\ & & 0 & & \end{array}$$

where f exists because $I(R_R)$ is injective . But $fi = \phi$ epic implies that f is epic i.e. $I(R_R) \in Gen(P_R)$.

(5) $\Rightarrow$ (3) Now $I(R_R) \in Gen(P_R)$ and then we have an epimorphism

$$P^{(X)} \xrightarrow{g} I(R_R) \longrightarrow 0$$

and a commutative diagram

$$\begin{array}{ccccc} R & \xleftarrow{=} & R & & \\ \exists h\big\downarrow & & \big\downarrow inc & & \\ P^{(X)} & \xrightarrow{g} & I(R_R) & \longrightarrow & 0 \end{array}$$

here h exists because R is projective and now $gh = inc$ implies that h is monic . But R_R is a cyclic module and so we get a monomorphism

$$0 \longrightarrow R_R \longrightarrow P_R^n$$

for some $n \in \mathbb{N}$.

Next we want to show that P_R generates every injective module $E_R \in$ *Mod-R* . Let $R_R^{(X)} \xrightarrow{\phi} E \longrightarrow 0$ be an epimorphism and then we have a monomorphism

$$0 \longrightarrow R_R^{(X)} \xrightarrow{i} P_R^{(Y)}$$

for some set Y . Consider the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & R^{(X)} & \xrightarrow{i} & P_R^{(Y)} \\ & & \phi\downarrow & & \downarrow \exists f \\ & & E & \xrightarrow{=} & E \\ & & \downarrow & & \\ & & 0 & & \end{array}$$

and here f exists because E_R is injective and $fi = \phi$ epic implies that f is epic i.e. $E \in Gen(P_R)$.

Now P_R generates every injective module and is a $*$-module , by proposition 3.4.5 , $Gen(P_R) = P^{\perp}$.

(3) $\Rightarrow$ (1) Since $P^{\perp} = Gen(P_R)$ which is closed under direct sums and factors so we have (i) P_R is finitely presented and $proj.dim(P_R) \leq 1$ by proposition 3.4.4 and (ii) $Ext^1_R(P,P) = 0$ since $P \in Gen(P_R) = P^{\perp}$.

Note that conditions (3) , (4) and (5) are equivalent and by (4) , we can let ${}_AP = {}_A\langle\, p_1, ..., p_n \,\rangle$ and then define the map

$$i : R \longrightarrow P^n \quad [r \mapsto (p_1r, ..., p_nr)]$$

. Here i is monic because if $(p_1r, ..., p_nr) = 0$ then $Pr = 0$ and hence $r = 0$ since P_R is faithful . And now we consider the exact sequence

$$(ex1): \quad 0 \rightarrow R \xrightarrow{i} P^n \rightarrow P^n/R \rightarrow 0$$

where $i(1) = (p_1, ..., p_n)$. We claim that P^n/R is a direct summand of $P^{(X)}$ for some set X .

By condition (4) , P_R is a $*$-module and $P^n/R \in Gen(P_R) = Pres(P_R) = P^{\perp}$, so we get an exact sequence

$$(ex2): \quad 0 \rightarrow L \rightarrow P^{(X)} \rightarrow P^n/R \rightarrow 0$$

where $L \in Gen(P_R) = P^{\perp}$. We want $(ex2)$ splits so P^n/R is a direct summand of $P^{(X)}$. But for this , it suffices to prove that $Ext^1_R(P^n/R, L) = 0$ [R

, corollary 5.2.39'] .

Apply $Hom_R(-, L)$ to $(ex1)$, we get the exact sequence

$$0 \longrightarrow Hom_R(P^n/R, L) \longrightarrow Hom_R(P^n, L) \xrightarrow{i^*} Hom_R(R, L) \cong L$$

$$\xrightarrow{\delta} Ext^1_R(P^n/R, L) \longrightarrow Ext^1_R(P^n, L) = 0$$

where $Ext^1_R(P^n, L) = 0$ because $L \in P^\perp$ and so δ is epic .

For $Ker(\delta) = Im(i^*) \leq Hom_R(R, L)$ we prove that i^* is epic and hence $Ext^1_R(P^n/R, L) = 0$.

Let $g \in Hom_R(R, L)$ such that $0 \neq x = g(1) \in L$. As $L \in Gen(P_R)$ there is an epimorphism $f : P^{(Y)} \longrightarrow L \longrightarrow 0$ for some set Y . So there is a $(q_j)_{j\in Y} \in P^{(Y)}$ such that $f((q_j)_{j\in Y}) = x = g(1)$ and the set $F \overset{def}{=} \{j \in Y \mid q_j \neq 0\}$ is finite . For each j ,

$$q_j \in {}_AP = {}_A\langle\, p_1, ..., p_n \,\rangle \text{ and hence } q_j = \sum_{i=1}^{n} a_i^j p_i$$

where $a_i^j \in Hom_R(P, P) = A$. For $j \in F$, we define

$$\phi_j = \bigoplus_{i=1}^{n} a_i^j : P^n \longrightarrow P \quad [(p_1, ..., p_n) \mapsto \sum_{i=1}^{n} a_i^j p_i]$$

and for $j \in Y \setminus F$, $\phi_j = 0$. Then we define

$$\phi : P^n \longrightarrow P^{(Y)} \text{ by } [x \mapsto (\phi_j(x))_{j\in Y}]$$

and so

$$\phi((p_1, ..., p_n)) = (\phi_j((p_1, ..., p_n)))_{j\in Y} = (q_j)_{j\in Y}$$

. Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "i"] \arrow[d, "g"'] & P^n \arrow[d, "\phi"] \\ & L & P^{(Y)} \arrow[l, "f"'] \end{tikzcd}$$

and $f\phi i(1) = f\phi((p_1, ..., p_n)) = f((q_j)_{j\in Y}) = g(1)$. Thus $g = f\phi i = i^*(f\phi)$ and $f\phi \in Hom_R(P^n, L)$. Therefore i^* is epic . □

3.4.8 It should be noted that in the proof of (3) $\Rightarrow$ (1) of the above theorem , we have the exact sequence

$$0 \longrightarrow R \longrightarrow P^n \longrightarrow P^n/R \longrightarrow 0$$

for some $n \in \mathbb{N}$ and P^n/R is a direct summand of $P^{(X)}$ for some X . But P^n/R is a finitely generated R-module , so actually it is a direct summand of a direct sum of *finitely many* copies of P i.e. we have an exact sequence

$$0 \longrightarrow R \longrightarrow P' \longrightarrow P'' \longrightarrow 0$$

with P' , $P'' \in add(P_R)$. Hence we may restate the definition of a tilting module , P_R , as a module satisfying the following conditions :

(1) There is an exact sequence $0 \longrightarrow R \longrightarrow P' \longrightarrow P'' \longrightarrow 0$ such that P' and P'' are in $add(V)$.

(2) $Ext^1_R(P, P) = 0$.

(3) $projdim(P) \leq 1$.

(4) P is finitely presented .

The following corollary shows the relationship between tilting modules and quasi-progenerators (or progenerators) .

Corollary 3.4.9 ([C2 , corollary 4] and [MO , proposition 4.4]) *Let P_R be a tilting module . Then the following conditions are equivalent :*

(1) P_R *is a selfgenerator .*

(2) P_R *is quasi-projective .*

(3) P_R *is a quasi-progenerator .*

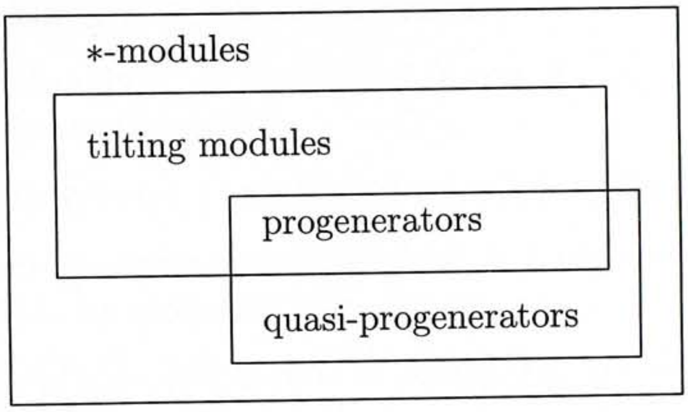
(4) P_R *is a progenerator .*

(5) P_R *is projective .*

(6) P_R *is a generator .*

Proof :
Assume condition (1) or (2) , by theorem 3.1.8 part (a) , we have (3) P_R is a quasi-progenerator and $Gen(P_R) = \overline{Gen}(P_R)$. But now P_R be a tilting module and so P_R generates every injective right R-module , hence , $Gen(P_R) = Mod\text{-}R$. Therefore we have (4) P_R is a progenerator . □

Now we can summarize the relations between $*$-modules , tilting modules , quasi-progenerators and progenerators here .



And the inclusions are proper because we have the following examples :

Example 3.4.10 Let P_R be a non-projective tilting module (for an example , see [HR , pp.126-127]) , then by corollary 3.4.8 , P_R is a $*$-module but NOT a quasi-progenerator . So if we let $A = End(P_R)$, $K_A = Hom_R(P, Q)$ for a cogenerator Q_R , then we have an equivalence

$$(- \otimes_A P : Cogen(K_A) \rightleftarrows Gen(P_R) : Hom_R(P, -)$$

but $Cogen(K_A) \neq Mod\text{-}A$.

Example 3.4.11 [CM , example 3.5] Let k be a field , and let R be a hereditary finite dimensional NOT semisimple k-algebra , e.g. the ring of n-dimensional upper triangular matrices over k . Then R_R is hereditary , artinian and the minimal injective cogenerator of $Mod\text{-}R$ is finitely generated . Moreover , if we let $(S_i)_{i=1}^m$ be a representative system of the isomorphy class of simple right R-modules and

$$P_R = \bigoplus_{i=1}^{m} I(S_i)^{l_i} \quad , \quad 0 < l_i < \infty$$

, then P_R is a $*$-module such that $Gen(P_R)$ = category of all injective right R-modules , hence P_R is a tilting module . But R is NOT semisimple , so P_R is NOT a generator and it follows from corollary 3.4.9 that P_R is NOT a quasi-progenerator .

Example 3.4.12 $\mathbb{Z}/2\mathbb{Z}$, as a $\mathbb{Z}$-module , is a $\mathbb{Z}$-quasi-progenerator but $Ext^1_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \neq 0$. So this module is not tilting .

Example 3.4.13 [D , remark 2] Let k be an algebraically closed field . Then there are a finite-dimensional k-algebra A and a $*$-module ${}_AM$ which is neither a tilting module nor a quasi-progenerator .

Let P_R be a $*$-module , $A = End(P_R)$ and $K_A = Hom_R(P, Q)$ for a cogenerator Q_R then we have

$$Cogen(K_A) = \{L \in Mod\text{-}A \mid Tor_1^A(L, {}_AP) = 0\ \}$$

which clearly contains every projective right A-module and is closed under taking submodules . In addition , if P_R is tilting then

$$Gen(P_R) = \{M \in Mod\text{-}R \mid Ext^1_R(P_R, M) = 0\ \}$$

which contains every injective right R-module and is closed under taking quotients . So every tilting module P_R induces an equivalence between a category which contains every projective right A-module and is closed under taking submodules and a category which contains every injective right R-module and is closed under taking quotients . In fact , these properties of such an equivalence characterize the tilting modules .

Theorem 3.4.14 [C2 , proposition 7] *Let ${}_AP_R$ be a bimodule , $T_P = (- \otimes_A P)$ and $H_P = Hom_R(P, -)$, Q_R be a cogenerator and $K_A = H_P(Q)$. let $\mathcal{C}_A \subseteq Mod\text{-}A$ and $\mathcal{G}_R \subseteq Mod\text{-}R$ be full subcategories such that*

$\mathcal{C}_A$ contains all free right A-modules and is closed under submodules ,

$\mathcal{G}_R$ contains all injective right R-modules and is closed under quotients .

If $\mathcal{C}_A \underset{H_P}{\overset{T_P}{\rightleftarrows}} \mathcal{G}_R$ defines an equivalence , then P_R is a tilting module , $A \cong End(P_R)$ canonically ,

$$\mathcal{C}_A = Cogen(K_A) = \{L \in Mod\text{-}A \mid Tor_1^A(L, {}_AP) = 0\ \}$$

and

$$\mathcal{G}_R = Gen(P_R) = \{M \in Mod\text{-}R \mid Ext^1_R(P_R, M) = 0\ \}$$

Proof :
We have

$$A \cong Hom_A(A, A) \cong Hom_R(T_P(A), T_P(A)) \cong Hom_R(P, P)$$

where isomorphisms are canonical and hence $A \cong End(P_R)$ canonically .

For any set X , $A_A^{(X)} \in \mathcal{C}_A$ so

$$P_R^{(X)} \cong T_P(A_A^{(X)}) \in \mathcal{G}_R$$

and hence , $Gen(P_R) \subseteq \mathcal{G}_R$. Moreover $\mathcal{G}_R = Im(T_P) \subseteq Gen(P_R)$ and therefore $Gen(P_R) = \mathcal{G}_R$.

Now fix a congenerator Q_R and for any set X , $t_P(Q^X) \in Gen(P_R)$, thus

$$\begin{aligned} K_A^X = Hom_R(P, Q)^X &\cong Hom_R(P, Q^X) \\ &\cong Hom_R(P, t_P(Q^X)) \in \mathcal{C}_A \end{aligned}$$

, so $Cogen(K_A) \subseteq \mathcal{C}_A$. Also $\mathcal{C}_A = Im(H_P) \subseteq Cogen(K_A)$ and hence $\mathcal{C}_A = Cogen(K_A)$.

It follows that P_R is a $*$-module and by assumption $I(R_R) \in \mathcal{G}_R = Gen(P_R)$, so P_R is a tilting module by theorem 3.4.7(5) .

Chapter 4

Equivalences and Dualities

4.1 The Equivalence $\mathcal{P}_A \sim \mathcal{I}_R$

Let A and R be rings , and denote with $\mathcal{P}_A$ the category of all projective right A-modules and with $\mathcal{I}_R$ the category of all injective right R-modules . Here we study the equivalences between $\mathcal{P}_A$ and $\mathcal{I}_R$ and the results are due to Colpi also [C3 , 1993] . Colpi proves that such an equivalence is precisely induced by a module U_R which is a finitely generated injective cogenerator with $A \cong End(P_R)$ canonically and R_R is artinian .

On the other hand , Azumaya have proved that a Morita duality between *A-mod* and *mod-R* , here *A-mod* (*mod-R*) is the category of all finitely generated left A-modules (right R-modules) , is also precisely induced by such a module ${}_AU_R$. So we have a good connection between equivalences and dualities in this situation .

First we give the following representation theorem due to Colpi [C3 , 1993] :

Theorem 4.1.1 [C3 , theorem 1] *Let A and R be rings . If*

$$T : \mathcal{P}_A \rightleftarrows \mathcal{I}_R : H$$

define a category equivalence then

(1) *For $U_R = T(A_A)$, $A \cong End(U_R)$ canonically .*

(2) *$T \cong (- \otimes_A U_R) = T_U$ and $H \cong Hom_R({}_AU_R, -) = H_U$ and*

(3) R_R *is artinian and* U_R *is a finitely generated injective cogenerator* .

Proof :
(1) If $U_R \overset{def}{=} T(A_A) \in \mathcal{I}_R$, then U_R has a canonical bimodule structure ${}_AU_R$ defined by the composition of the ring homomorphisms

$$A \xrightarrow{cano} End(A_A) \xrightarrow{T} End(T(A)_R) = End(U_R) \quad .$$

But now T is an equivalence , so in fact this is a ring isomorphism and hence $A \cong End(U_R)$ canonically .

(2) Note that now (T, H) is an adjoint pair with respect to $\mathcal{P}_A$ and $\mathcal{I}_R$ and $A_A \in \mathcal{P}_A$. So for every module $M \in \mathcal{I}_R$, we have the isomorphisms

$$H(M) \cong Hom_A(A, H(M)) \cong Hom_R(T(A), M)$$

which are natural in M and hence $H \cong Hom_R({}_AU_R, -)$.

We need to prove that $T \cong T_U$ and it suffices to show that $T_U(N) \in \mathcal{I}_R$, for every $N \in \mathcal{P}_A$.
(i) First we assert that $\mathcal{I}_R$ is closed under taking direct sums .

Recall that for a family of objects $\{L_\lambda\}_\Lambda$ of a category $\mathcal{C}$, an object K in $\mathcal{C}$ with morphisms $\{\varepsilon_\lambda : L_\lambda \longrightarrow K\}_\Lambda$ is called the coproduct of the family $\{L_\lambda\}_\Lambda$ if for every family of morphisms $\{g_\lambda : L_\lambda \longrightarrow Y\}_\Lambda$ in $\mathcal{C}$, there is a unique morphism $g : K \longrightarrow Y$ such that $g \circ \varepsilon_\lambda = g_\lambda$. [W , 9.5]

Clearly the category $\mathcal{P}_A$ has coproducts which are exactly the direct sums of projective modules .

Now let $\{M_\lambda\}_\Lambda$ in $\mathcal{I}_R$ and then for each λ , we get a $N_\lambda \in \mathcal{P}_A$ such that $T(N_\lambda) = M_\lambda$. Let

$$M = \bigoplus_{\lambda\in\Lambda} M_\lambda \ \text{ and } \ i_\lambda : M_\lambda \longrightarrow M \ \text{ be the inclusion ,}$$

$$N = \bigoplus_{\lambda\in\Lambda} N_\lambda \in \mathcal{P}_A \ \text{ and } \ j_\lambda : N_\lambda \longrightarrow N \ \text{ be the inclusion .}$$

then $\{j_\lambda : N_\lambda \longrightarrow N\}_\Lambda$ is a coproduct for $\{N_\lambda\}_\Lambda$ in $\mathcal{P}_A$.

Since T is an equivalence , the family $\{T(j_\lambda) : T(N_\lambda) \longrightarrow T(N)\}_\Lambda$ is a coproduct for $\{T(N_\lambda)\}_\Lambda$ in $\mathcal{I}_R$.

For every λ , consider $T(N_\lambda) \xrightarrow{i_\lambda} M \xrightarrow{inc} I(M)$ and we get a commutative diagram in $\mathcal{I}_R$,

$$\begin{array}{ccc} T(N_\lambda) & \xrightarrow{T(j_\lambda)} & T(N) \\ {\scriptstyle inc \circ i_\lambda}\downarrow & & \downarrow{\scriptstyle \exists\ \beta} \\ I(M) & \xrightarrow{=} & I(M) \end{array}$$

. On the other hand , if we let

$$\alpha = \bigoplus_{\lambda\in\Lambda} T(j_\lambda) : M \longrightarrow T(N)$$

then

$$\beta \circ \alpha \circ i_\lambda = \beta \circ T(j_\lambda) = i_\lambda$$

for every λ . Therefore $\beta \circ \alpha = Id_M$ and so $M \cong \alpha(M)$ is a direct summand of $T(N)$, hence injective .

(ii) Now we can prove that $T \cong (- \otimes_A U_R) = T_U$ naturally . First we show that $T_U(\mathcal{P}_A) \subseteq \mathcal{I}_R$ i.e. $T_U : \mathcal{P}_A \longrightarrow \mathcal{I}_R$ is a well defined functor .

To see this , let $N \in \mathcal{P}_A$ and so we get $N \bigoplus K = A_A^{(X)}$ for some set X . Since an additive functor preserves split exact sequences , $T_U(N)$ is canonically a direct summand of

$$T_U(A_A^{(X)}) = A_A^{(X)} \otimes_A U_R \cong U_R^{(X)} \in \mathcal{I}_R$$

thus $T_U(N) \in \mathcal{I}_R$ also .

Now both T and T_U are right adjoints to $H \cong H_U : \mathcal{I}_R \longrightarrow \mathcal{P}_A$, it follows that $T \cong T_U$ naturally .

(3) (i) The module U_R is finitely generated . Suppose NOT , then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of elements in U_R such that

$$x_n \in U_R \setminus \langle\, x_1, ..., x_{n-1} \,\rangle_R \ \text{ for all } n \geq 2 \,.$$

Let $L = \langle\, x_n \mid n \in \mathbb{N} \,\rangle_R \leq U_R$ and Q_R be an injective cogenerator of $Mod\text{-}R$.

For $n \geq 2$, let $\nu_n : L \longrightarrow L/(\langle\, x_1, ..., x_{n-1} \,\rangle_R)$ be the natural epimorphism and $\psi_n : L/(\langle\, x_1, ..., x_{n-1} \,\rangle_R) \longrightarrow Q_R$ be a morphism such that

$$\psi_n(x_n + \langle\, x_1, ..., x_{n-1} \,\rangle_R) \neq 0$$

, then we get $\phi_n \overset{def}{=} \psi_n \circ \nu_n : L \longrightarrow Q$ and

$$\phi_n(x_n) \neq 0 \quad \text{but} \quad \phi(x_j) = 0 \quad \text{for} \quad 1 \leq j \leq n-1$$

. Next we define $\phi : L \longrightarrow Q^{(\mathbb{N})}$ by $\phi(z) = (\phi_n(z))_{n\in\mathbb{N}}$ for every $z \in L$. ϕ is well defined because if we write $z = \sum_{k=1}^{N} x_k r_k$ where $r_k \in R$ and $N \geq 2$ then $\phi_j(x_k) = 0$ for all $j > N$ and $1 \leq k \leq N$, hence $\phi_j(z) = 0$ for all $j > N$. So $\phi(L) \subseteq Q^{(\mathbb{N})}$.

Let $\pi_n : Q^{(\mathbb{N})} \longrightarrow Q$ be the n-th surjection then for every n , $\pi_n \circ \phi(x_n) \neq 0$.

Since $\mathcal{I}_R$ is closed under direct sums so $Q^{(\mathbb{N})}$ is injective and we get the commutative diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & L & \xrightarrow{inc} & U_R \\
 & & \phi\big\downarrow & & \big\downarrow \exists f \\
 & & Q^{(\mathbb{N})} & \xrightarrow{=} & Q^{(\mathbb{N})}
\end{array}$$

so that $f \circ inc = \phi$ and then for every n , $\pi_n \circ f(x_n) \neq 0$.

On the other hand

$$\begin{array}{rcl}
Hom_R(U, Q^{(\mathbb{N})}) & \overset{(\rho_Q^{(\mathbb{N})})_*}{\cong} & H_U[(T_U H_U(Q))^{(\mathbb{N})}] \\
 & \overset{\mu_*}{\cong} & H_U T_U(H_U(Q)^{(\mathbb{N})}) \\
 & \overset{\sigma}{\cong} & H_U(Q)^{(\mathbb{N})} = Hom_R(U, Q)^{(\mathbb{N})}
\end{array}$$

so for $f \in Hom_R(U, Q^{(\mathbb{N})})$, there exists a $\varphi = (\varphi_n)_{n\in\mathbb{N}} \in Hom_R(U, Q)^{(\mathbb{N})}$ such that

$$f = [(\rho_Q^{(\mathbb{N})})_* \circ \mu_* \circ \sigma](\varphi) : U \longrightarrow Q^{(\mathbb{N})}$$

i.e. $f = \rho_Q^{(\mathbb{N})} \circ \mu \circ (\sigma(\varphi))$.

Let $u \in U_R$,

$$\begin{array}{rcl}
f(u) & = & \rho_Q^{(\mathbb{N})} \circ \mu \circ (\sigma(\varphi))(u) \\
 & = & \rho_Q^{(\mathbb{N})}(\mu((\varphi_n)_{n\in\mathbb{N}} \otimes u))) \\
 & = & \rho_Q^{(\mathbb{N})}((\varphi_n \otimes u)_{n\in\mathbb{N}}) \\
 & = & (\rho_Q(\varphi_n \otimes u))_{n\in\mathbb{N}} \\
 & = & (\varphi_n(u))_{n\in\mathbb{N}}
\end{array}$$

. However if we fix some integer $m > max\{n \mid \varphi_n \neq 0\}$ and for $x_m \in U_R$,

$$0 \neq \pi_m(f(x_m)) = \pi_m((\varphi_n(x_m))_{n \in \mathbb{N}}) = 0$$

this is a contradition .
(ii) R_R is noetherian as $\mathcal{I}_R$ is closed under direct sums [AF , 18.13] .
(iii) We claim that now every injective right R-module is a direct summand of a direct sum of copies of U_R .

For if $M \in \mathcal{I}_R$, then $H_U(M) \in \mathcal{P}_A$ and so $H_U(M)$ is a direct summand of $A_A^{(X)}$ for some set X . Hence $M \cong T_U H_U(M)$ is a direct summand of $T_U(A_A^{(X)}) \cong U_R^{(X)}$.

In addition if $0 \neq M_R$ is an indecomposable injective module then since R_R is noetherian , we can write $U = \bigoplus_\Lambda U_\alpha$ where U_α are indecomposable (injective) modules [AF , 25.6] and so $U^{(X)} = (\bigoplus U_\alpha)^{(X)}$.

On the other hand , as M is a direct summand of $U^{(X)}$, we have another indecomposable decomposition $U^{(X)} = M \bigoplus (\bigoplus_B L_\beta)$. Note that the endomoephism ring of every indecomposable injective module is local [AF , lemma 25.4] and again by [AF , 12.6 Azumaya theorem] , these two indecomposable decompositions are equivalent , so $M \cong U_\alpha$ for some α . It follows that every indecomposable injective right R-module is a direct summand of U_R .

Moreover every indecomposable injective right R-module is finitely generated as U_R is finitely generated .

(iv) We claim that if R_R is noetherian then any finitely generated right R-module M_R has an injective hull $I(M)$ which is a direct sum of finite number of indecomposable injective modules . See also [AF , theorem 25.6] .

To prove this , let $M \leq I(M)$ where $I(M)$ is the injective hull of M . Since R_R is noetherian , $I(M) = \bigoplus_{j \in J} K_j$ where K_j are indecomposable (injective) modules [AF 25.5] . Hence $\{K_j\}_{j \in J}$ is a set of independent submodules of $I(M)$ and then $\{K_j \cap M\}_{j \in J}$ is a set of independent submodules of M_R . But M_R is noetherian because R_R is noetherian and M_R is finitely generated . Thus all but finitely many $K_j \cap M$ are zero . As M_R is an essential submodule of $I(M)$, $K_j \cap M = 0$ implies that $K_j = 0$. Now we have all but finitely many K_j are zero i.e. $I(M)$ is a direct sum of finite number of indecomposable injective modules .

(v) Therfore for any finitely generated right R-module M_R , its injective hull $I(M)$, By (iii) and (iv) , is also finitely generated . Hence by [Fa , corollary 20.13] , R_R is artinian .
(vi) Finally U_R is an (injective) cogenerator since U_R cogenerates $I(S)$ for every simple right R-module S_R . Or one can check that actually $I(S)$ is indecomposable and hence is a direct summand of U_R . □

Conversely we have the following

Theorem 4.1.2 [C3 , theorem 1] *Suppose that R_R is artinian with a finitely generated injective cogenerator U_R such that $A = End(U_R)$. Let $T_U = (-\otimes_A U_R)$ and $H_U = Hom_R({}_AU_R, -)$ with their associated natural morphisms σ and ρ , then*

$$T_U : \mathcal{P}_A \rightleftarrows \mathcal{I}_R : H_U$$

defines an inverse category equivalence between $\mathcal{P}_A$ and $\mathcal{I}_R$.

Proof :
Now $T_U(A_A^{(X)}) \cong U_R^{(X)}$ and as U_R is finitely generated , we get

$$H_U(U_R^{(X)} \cong Hom_R(U, U_R)^{(X)} = A_A^{(X)}$$

. So if $N \in \mathcal{P}_A$ then N is a direct summand of $A_A^{(X)}$ where X is a set . Hence $T_U(N)$ is a direct summand of $T_U(A_A^{(X)}) \cong U_R^{(X)}$ which is injective because U_R is injective and R_R is artinian (noetherian) [AF , 18.13] .

On the other hand , for $M \in \mathcal{I}_R$, since U_R is an injective cogenerator we get an inclusion $0 \longrightarrow M \longrightarrow U_R^X$ for some set X .

We claim that $M \subseteq U_R^{(X)}$. To see this , let $z \in M_R$ then $zR \leq U_R^X$. But zR is finitely generated and R_R is artinian , by [AF , 10.18] , there is a finite set $F \subseteq X$ such that

$$zR \leq U_R^F \leq U_R^{(X)}$$

. So $z \in U_R^{(X)}$.

Now $M_R \leq U_R^{(X)}$ and since M_R is injective , it is a direct summand of $U_R^{(X)}$. Hence $H_U(M)$ is a direct summand of $H_U(U_R^{(X)}) \cong A_A^{(X)}$ which is projective . Thus we have $H_U(M) \in \mathcal{P}_A$.

We have proved that

$$T_U : \mathcal{P}_A \longrightarrow \mathcal{I}_R \text{ and } H_U : \mathcal{I}_R \longrightarrow \mathcal{P}_A$$

are well defined .

Let $N \in \mathcal{P}_A$ we need to prove that σ_N is an isomorphism .

(i) Since N_A is a direct summand of $A_A^{(X)}$ for some set X , so $A_A = Hom_R(U, U_R)$ cogenerates N_A and here U_R is an (injective) cogenerator of $Mod\text{-}R$. Hence , by proposition 1.1.4(3) , σ_N is monic .

(ii) First we note that $\sigma_{A^{(X)}}$ is an isomorphism for every set X as

$$\sigma_{A^{(X)}} : A_A^{(X)} \cong H_U(U_R^{(X)}) \cong H_U T_U(A^{(X)})$$

here all isomorphisms are canonical . Since N_A is a direct summand of $A_A^{(X)}$ for some set X , we have a *split epimorphism*

$$A_A^{(X)} \to N \to 0$$

and applying the additive functor $H_U T_U$, we get the commutative diagram with split exact rows

$$\begin{array}{ccccc} A_A^{(X)} & \longrightarrow & N & \longrightarrow & 0 \\ \Big\downarrow{\scriptstyle \sigma_{A^{(X)}}} & & \Big\downarrow{\scriptstyle \sigma_N} & & \\ H_U T_U(A_A^{(X)}) & \longrightarrow & H_U T_U(N) & \longrightarrow & 0 \end{array}$$

, and from this diagram , σ_N is epic .

(a) Now for $M \in \mathcal{I}_R$, since M_R is a direct summand of $U_R^{(X)}$ for some set X , so U_R generates M_R and hence ρ_M is epic by proposition 1.1.4(4) .

(b) Consider the *split monomorphism*

$$0 \to M \to U_R^{(X)}$$

and applying the additive functor $T_U H_U$, we get the commutative diagram with split exact rows

$$\begin{array}{ccccc} 0 & \longrightarrow & T_U H_U(M) & \longrightarrow & T_U H_U(U_R^{(X)}) \\ & & \Big\downarrow{\scriptstyle \rho_M} & & \Big\downarrow{\scriptstyle \rho_{U^{(X)}}} \\ 0 & \longrightarrow & M & \longrightarrow & U_R^{(X)} \end{array}$$

, and from this diagram , ρ_M is monic . □

Let R-*mod* (*mod*-R) be the category of all finitely generated left (right) R-modules . It is well known that for a bimodule ${}_AU_R$, the functors

$$Hom_A(-,\ {}_AU_R) : A\text{-}Mod \to Mod\text{-}R \text{ and}$$

$$Hom_R(-,\ {}_AU_R) : Mod\text{-}R \to A\text{-}Mod$$

induce a duality between the subcategories A-*mod* and *mod*-R if and only if R_R is artinian , U_R is a finitely generated injective cogenerator and $A = End(U_R)$. Moreover in this situation , ${}_AA$ is artinian and ${}_AU$ is a finitely generated injective cogenerator , and ${}_AU_R$ is faithfully balanced . Furthermore this happens if and only if ${}_AA$ and R_R are artinian , Morita dual rings . The reader is referred to [Fa] and [AF] for these . Now we can restate the above theorems as :

Theorem 4.1.3 [C3 , theorem 2] *Let A and R be rings , ${}_AU_R$ be a bimodule . Then the following conditions are equivalent :*

(1) *there exists an equivalence* $\mathcal{P}_A \underset{H}{\overset{T}{\rightleftarrows}} \mathcal{I}_R$.

(2) *there exists an equivalence* ${}_R\mathcal{P} \underset{H'}{\overset{T'}{\rightleftarrows}} {}_A\mathcal{I}$.

(3) *there exists a duality* A-*mod* $\underset{H_2}{\overset{H_1}{\rightleftarrows}}$ *mod*-R .

(4) ${}_AA$ *and* R_R *are artinian , Morita dual rings .*

(5) R_R *is artinian ,* U_R *is a finitely generated injective cogenerator and* $A = End(U_R)$.

(6) ${}_AA$ *is artinian and* ${}_AU$ *is a finitely generated injective cogenerator and* $R = End({}_AU)$.

If the previous conditions are satisfied , then there are natural isomorphisms of functors :

$$T \cong (- \otimes_A U_R) \text{ and } H \cong Hom_R({}_AU_R, -) \ ,$$

$$T' \cong ({}_AU \otimes_R -) \text{ and } H' \cong Hom_A({}_AU_R, -) \ ,$$

$$H_1 \cong Hom_A(-,\ {}_AU_R) \text{ and } H_2 \cong Hom_R(-,\ {}_AU_R) \ .$$

Consequently we have the well known characterizations of a quasi-Frobenius ring (QF-ring) .

Corollary 4.1.4 *The following conditions on a ring R are equivalent :*

(a) R is quasi-Frobenius .

(b) $\mathcal{P}_R = \mathcal{I}_R$.

(c) ${}_R\mathcal{P} = {}_R\mathcal{I}$.

Proof :
R is an QF-ring iff R_R is artinian and R_R is an injective cogenerator . By the above theorem , for $A = R$ and $U_R = R_R$, this happens iff $(- \otimes_R R_R) \cong Id_{Mod\text{-}R}$ and $Hom_R({}_RR_R, -) \cong Id_{Mod\text{-}R}$ induce an equivalence between $\mathcal{P}_R$ and $\mathcal{I}_R$. Clearly this means that $\mathcal{P}_R = \mathcal{I}_R$ Note that we always have $R \cong End(R_R)$ canonically .

4.2 The Equivalence $FGP\text{-}A \sim FCI\text{-}R$

For a ring R , we denote $FGP\text{-}R$ ($FCI\text{-}R$) the category of all finitely generated projective (finitely cogenerated injective) right R-modules . Similarly we introduce the notations $R\text{-}FGP$ and $R\text{-}FCI$ for left R-modules .

By Morita theorem , an equivalence between $Mod\text{-}A$ and $Mod\text{-}R$ for two rings A and R is precisely induced by a bimodule ${}_AP_R$ where P_R is a finitely generated projective generator i.e. progenerator and $A \cong End(P_R)$ canonically .

In [X , 1995] , Xue proved that an equivalence between $FGP\text{-}A$ and $FCI\text{-}R$ is precisely induced by a bimodule ${}_AU_R$ where U_R is a finitely cogenerated injective cogenerator and $A \cong End(U_R)$ canonically , also this happens if and only if ${}_AU_R$ induces a duality between $A\text{-}FGP$ and $FCI\text{-}R$ i.e. the functors

$$H_1 = Hom_A(-, {}_AU_R) : A\text{-}FGP \rightleftarrows FCI\text{-}R : Hom_R(-, {}_AU_R) = H_2$$

define a duality .

We first give a representation theorem .

Theorem 4.2.1 [X ,theorem 2] *Let $T : FGP\text{-}A \rightleftarrows FCI\text{-}R : H$ define an (additive) equivalence , and ${}_AU_R = T({}_AA_A)$ be the canonical bimodule . Then*

(1) U_R is a finitely cogenerated injective cogenerator and $A \cong End(U_R)$ canonically .

(2) $T \cong (- \otimes_A U_R)$ and $H \cong Hom_R({}_AU_R, -)$.

Proof :
Since T is an equivalence , for the canonical bimodule ${}_AU_R = T({}_AA_A)$, we have $A \cong End(U_R)$ canonically .

As (T, H) is an adjoint pair with respect to $FGP\text{-}A$ and $FCI\text{-}R$, so for $E_R \in FCI\text{-}R$, we get the isomorphisms

$$H(E)_A \cong Hom_A(A, H(E)_A) \cong Hom_R(T(A), E) = Hom_R({}_AU_R, E)$$

which are natural in $E \in FCI\text{-}R$. Hence $H \cong Hom_R({}_AU_R, -)$.

U_R is finitely cogenerated injective because $U = T(A) \in FCI\text{-}R$.

Now we show that U_R is a cogenerator .

Let $E_R = E(S_R)$ be the injective envelope of a simple R-module S_R . Then , for $E \in FCI\text{-}R$ [AF , 18.18] , $H(E) \in FGP\text{-}A$. So we have

$$A_A^n \cong H(E) \bigoplus K$$

for some $K \in FGP\text{-}A$ and $n \in \mathbb{N}$ [AF , 17.3] . Hence , by applying T ,

$$U_R^n \cong T(A_A^n) \cong TH(E) \bigoplus T(K) \cong E \bigoplus T(K)$$

and so U_R cogenerates $E_R = E(S_R)$. Thus U_R is a (injective) cogenerator [AF , 18.15] .

Now U_R is a finitely cogenerated injective cogenerator and we can check that $T_U = (- \otimes_A U_R) : FGP\text{-}A \to FCI\text{-}R$ is a well defined functor . If $P \in FGP\text{-}A$ then by [AF , 17.3] ,

$$P \bigoplus Q \cong A_A^n$$

for some $Q \in FGP\text{-}A$ and $n \in \mathbb{N}$. Hence

$$(P \otimes_A U_R) \bigoplus (Q \otimes_A U_R) \cong (P \bigoplus Q) \otimes_A U_R \cong A_A^n \otimes_A U_R \cong U_R^n$$

and so $(P \otimes_A U_R) \in FCI\text{-}R$ by [AF , 10.8] .

Finally since both T and T_U are inverse functors of $H \cong H_U$, we have that $T \cong T_U$.

Now we can give the main theorem about the equivalence $FGP\text{-}A \sim FCI\text{-}R$.

Theorem 4.2.2 [X , theorem 1] *Let $_AU_R$ be a bimodule . Then the following statements are equivalent :*

(1) U_R is a finitely cogenerated injective cogenerator and $A \cong End(U_R)$ canonically .

(2) $T_U = (- \otimes_A U_R) : FGP\text{-}A \rightleftarrows FCI\text{-}R : Hom_R({}_AU_R, -) = H_U$ define an equivalence .

(3) $H_1 = Hom_A(-, {}_AU_R) : A\text{-}FGP \rightleftarrows FCI\text{-}R : Hom_R(-, {}_AU_R) = H_2$ define a duality .

Proof :
(1) $\Rightarrow$ (2) (i) If $P \in FGP\text{-}A$ then

$$P \bigoplus Q \cong A_A^n$$

for some $Q \in FGP\text{-}A$ and $n \in \mathbb{N}$, by [AF , 17.3] . Hence

$$(P \otimes_A U_R) \bigoplus (Q \otimes_A U_R) \cong (P \bigoplus Q) \otimes_A U_R \cong A_A^n \otimes_A U_R \cong U_R^n$$

and so $(P \otimes_A U_R) \in FCI\text{-}R$ by [AF , 10.8] .
(ii) Let $E \in FCI\text{-}R$, as U_R is a cogenerator , we have

$$E \bigoplus F \cong U_R^n$$

for some $F \in FCI\text{-}R$ and $n \in \mathbb{N}$. Hence

$$Hom_R(U_R, E) \bigoplus Hom_R(U_R, F) \cong Hom_R(U_R, U_R^n) \cong A_A^n$$

and so $Hom_R(U_R, E) \in FGP\text{-}A$.
Now we have that

$$T_U = (- \otimes_A U_R) : FGP\text{-}A \rightleftarrows FCI\text{-}R : Hom_R({}_AU_R, -) = H_U$$

are well defined functors .
(iii) Note that for every $n \in \mathbb{N}$, we have the canonical isomorphisms :

$$A^n \otimes_A U_R \cong U_R^n \text{ and } Hom_R(U, U_R^n) \cong A_A^n \ .$$

(iv) Let $P_A \in FGP\text{-}A$ and we claim that the canonical homomorphism

$$\sigma_P : P \longrightarrow Hom_R(U, P \otimes_A U) \quad [p \mapsto [u \mapsto p \otimes u]]$$

is an isomorphism . For simplicity , we may assume $A = End(U_R)$ because $A \cong End(U_R)$ canonically . And then

$$Hom_A(P, A_A) = Hom_A(P, Hom_R(U,U)) \overset{\Phi}{\cong} Hom_R(P \otimes_A U, U)$$

where Φ = adjoint isomorphism . Since A_A cogenerates P_A and so if $0 \neq p \in P$, there is a $h \in Hom_A(P, A)$ such that $h(p) \neq 0$. Now $h(p) \in A = End(U_R)$, hence there exists a $u \in U$ such that $0 \neq h(p)(u) = \Phi(h)(p \otimes u)$ which implies that $0 \neq p \otimes u = \sigma_P(p)(u)$ i.e. $\sigma_P(p) \neq 0$. And we get that σ_P is monic .

As $P \in FGP\text{-}A$, $P \bigoplus Q \cong A_A^n$ for some $Q \in FGP\text{-}A$ and $n \in \mathbb{N}$ and by (iii) , σ_{A^n} is an isomorphism . For the split epimorphism $A_A^n \longrightarrow P_A \longrightarrow 0$, by applying the additive functor $H_U T_U$, we get the commutative diagram with split exact rows :

$$\begin{tikzcd}
A_A^n \arrow[r] \arrow[d, "\sigma_{A^n}"'] & P_A \arrow[r] \arrow[d, "\sigma_P"] & 0 \\
H_U T_U(A_A^n) \arrow[r] & H_U T_U(P_A) \arrow[r] & 0
\end{tikzcd}$$

and from this diagram , we obtain that σ_P is epic .

(v) For any $E_R \in FCI\text{-}R$, $E \bigoplus F \cong U_R^n$ for some $F \in FCI\text{-}R$ and $n \in \mathbb{N}$. So $E \in Gen(U_R)$ and then ρ_E is epic . By (iii) , ρ_{U^n} is an isomorphism and for the split monomorphism $0 \longrightarrow E \longrightarrow U_R^n$, applying $T_U H_U$, we get the commutative diagram with split exact rows :

$$\begin{tikzcd}
0 \arrow[r] & T_U H_U(E) \arrow[r] \arrow[d, "\rho_E"'] & T_U H_U(U_R^n) \arrow[d, "\rho_{U^n}"] \\
0 \arrow[r] & E \arrow[r] & U_R^n
\end{tikzcd}$$

and from this diagram , we obtain that ρ_E is monic .

(2) $\Rightarrow$ (1) Since $A_A \in FGP\text{-}A$, $U_R \cong A \otimes_A U_R \in FCI\text{-}R$. And we have

$$A \overset{\sigma_A}{\cong} Hom_R(U, A \otimes_A U) \overset{\mu_*}{\cong} Hom_R(U, U)$$

where $\mu : A \otimes_A U \longrightarrow U$ $[a \otimes u \mapsto au]$ and $\mu_* = Hom_R(U, \mu)$. Clearly

$$\mu_* \circ \sigma_A : A \cong Hom_R(U, U) = End(U_R)$$

is the canonical ring isomorphism .

We need to prove that U_R is an (injective) cogenerator .

Let $E_R = E(S_R)$ be the injective envelope of a simple right R-module S_R , then $E_R = E(S_R) \in FCI\text{-}R$ and so $Hom_R(U, E) \in FGP\text{-}A$ i.e.

$$A_A^n \cong Hom_R(U, E) \bigoplus Q$$

for some $Q \in FGP\text{-}A$ and $n \in \mathbb{N}$. Then

$$U_R^n \cong A^n \otimes_A U_R \cong (Hom_R(U, E) \otimes_A U_R) \bigoplus (Q \otimes_A U_R) \cong E \bigoplus (Q \otimes_A U_R)$$

thus U_R cogenerates E_R and by [AF , 18.16] , U_R is an (injective) cogenerator .

(1) $\Rightarrow$ (3) Denote $(-)^* = Hom(-, U)$ and then $M^* = Hom_A(M, U)$ for $M \in A\text{-}Mod$ and $M^* = Hom_R(M, U)$ for $M \in Mod\text{-}R$. And a (left A-module) right R-module M is U-reflexive if the evaluation homomorphism

$$e_M : M \longrightarrow M^{**} \quad [m \mapsto [f \mapsto f(m)]]$$

is an isomorphism .

Now we have the canonical right R-module isomorphism

$$({}_AA)^* = Hom_A({}_AA, U) \cong U_R$$

and $A \cong End(U_R) = Hom_R(U, U)$ implies that $(U_R)^* \cong_A A$. Hence both ${}_AA$ and U_R are U-reflexive modules and so are ${}_AA^n$ and U_R^n for every $n \in \mathbb{N}$. [AF , 20.13]

For ${}_AP \in A\text{-}FGP$, then ${}_AA^n \cong P \bigoplus Q$ for some $Q \in A\text{-}FGP$ and $n \in \mathbb{N}$. By [AF , 20.13] , ${}_AP$ is U-reflexive i.e.

$$e_P : P \cong P^{**} \quad \text{for } {}_AP \in A\text{-}FGP$$

. Moreover since

$$U_R^n \cong ({}_AA^n)^* \cong P^* \bigoplus Q^*$$

, we get $P^* \in FCI\text{-}R$ i.e. $(-)^* : A\text{-}FGP \longrightarrow FCI\text{-}R$ is well defined .

On the other hand, for $E_R \in FCI\text{-}R$ then $U_R^n \cong E \bigoplus F$ for some $F \in FCI\text{-}R$ and $n \in \mathbb{N}$. Since U_R^n is U-reflexive, so is its direct summand E_R. Also note that

$${}_AA^n \cong (U_R^n)^* \cong E^* \bigoplus F^*$$

and we have $E^* \in A\text{-}FGP$ i.e. $(-)^* : FCI\text{-}R \longrightarrow A\text{-}FGP$ is well defined.

(3) $\Rightarrow$ (1) (i) Since ${}_AA \in A\text{-}FGP$, we get the canonical left A-module isomorphism

$${}_AA \cong ({}_AA)^{**} = Hom_R(Hom_A(A, U), U) \cong Hom_R(U, U)$$

which induces the canonical ring isomorphism $A \cong End(U_R)$.
(ii) For ${}_AA \in A\text{-}FGP$, $Hom_A(A, U) \cong U_R \in FCI\text{-}R$.
(iii) Finally we check that U_R is an (injective) cogenerator.

Let $E_R = E(S_R)$ be the injective envelope of a simple right R-module S_R, then $E_R = E(S_R) \in FCI\text{-}R$ and hence $E^* \in A\text{-}FGP$ and

$${}_AA^n \cong E^* \bigoplus Q$$

for some $Q \in A\text{-}FGP$ and $n \in \mathbb{N}$, thus

$$U_R^n \cong ({}_AA^n)^* \cong E^{**} \bigoplus Q^* \cong E \bigoplus Q^*$$

and so U_R cogenerates $E = E(S_R)$. Finally we obtain that U_R is an (injective) cogenerator. $\square$

We now see from the equivalence of (1) and (3) in the above theorem that a finitely cogenerated injective cogenerator U_R induces a duality between A-FGP and $FCI\text{-}R$, where $A \cong End(U_R)$ canonically. Dual to this theorem, actually any duality $A\text{-}FGP \rightleftarrows FCI\text{-}R$ is also induced by such a module U_R.

Theorem 4.2.3 [X, Theorem 3] *Let $H_1 : A\text{-}FGP \rightleftarrows FCI\text{-}R : H_2$ define a (additive) duality, and ${}_AU_R = H_1({}_AA_A)$ be the canonical bimodule. Then*

(a) U_R is a finitely cogenerated injective cogenerator and $A \cong End(U_R)$ canonically.

(b) $H_1 \cong Hom_A(-, {}_AU_R)$ and $H_2 \cong Hom_R(-, {}_AU_R)$

A ring R is called a *right (left) PF-ring* if R_R $({}_RR)$ is a finitely cogenerated injective cogenerator . We have the following similar result as corollary 4.1.4 :

Corollary 4.2.4 *A ring R is a right PF-ring if and only if we have an equality FGP-$R = FCI$-R .*

We remark that there is a right PF-ring which is not a left PF-ring [DM] .

Chapter 5

Torsion Theories Induced by Tilting Modules

5.1 The Tilting Theorem

Our results in the previous chapters are mostly stated for *right* modules, however, one may notice that, with minor change of notation, most of the theorems are true for *left* modules also. Therefore, although some results are originally proved for *right* modules, we may use the *left* version of the results from now on.

Recall that a module V is a tilting module provided the following conditions are satisfied:

(1) There is an exact sequence $0 \longrightarrow R \longrightarrow V' \longrightarrow V'' \longrightarrow 0$ such that V' and V'' are in $add(V)$.

(2) $Ext^1_R(V, V) = 0$.

(3) $projdim(V) \leq 1$.

(4) V is finitely presented.

Here we present the Tilting Theorem due to R. R. Colby and K. R. Fullar [CbF1, 1990] for tilting modules over a arbitrary ring. This theorem characterizes the tilting modules. First we fix our notation. For a left R-module ${}_RV$ and $B = End({}_RV)$, we let $H = Hom_R(V, -)$, $H' = Ext^1_R(V, -)$

be the derived functor of H , $T = (V \otimes_B -)$ and $T' = Tor_1^B(V, -)$ be the derived functor of T . Associated with T and H , we have the natural transformations

$$\rho : TH \longrightarrow Id_{R\text{-}Mod} \text{ and } \sigma : Id_{B\text{-}Mod} \longrightarrow HT$$

given by

$$\rho_M : V \otimes_B Hom_R(V, M) \longrightarrow M \quad [v \otimes f \mapsto f(v)]$$

$$\sigma_N : N \longrightarrow Hom_R(V, V \otimes_B N) \quad [n \mapsto [v \mapsto v \otimes n]]$$

for each $M \in R\text{-}Mod$ and $N \in B\text{-}Mod$.

Recall that if ${}_RV$ is a tilting module and $B = End({}_RV)$, then we have the equivalence :

$$T = (V \otimes_B -) : Cogen({}_BK) \rightleftarrows Gen({}_RV) : H = Hom_R(V, -)$$

, here ${}_BK = Hom_R(V, {}_RQ)$ and ${}_RQ$ is a cogenerator in $R\text{-}Mod$, i.e. ρ_M is an isomorphism for every $M \in Gen({}_RV)$ and σ_N is an isomorphism for every $N \in Cogen({}_BK)$. Moreover

$$Cogen({}_BK) = \{{}_BN \mid Tor_1^B(V, N) = 0\} = Ker(T') \text{ and}$$

$$Gen({}_RV) = \{{}_RM \mid Ext_R^1(V, M) = 0\} = Ker(H') \ .$$

Also we have the functors :

$$\Delta = Hom_R(-, V) : R\text{-}Mod \longrightarrow Mod\text{-}B \ ,$$

$$\Delta = Hom_B(-, V) : Mod\text{-}B \longrightarrow R\text{-}Mod \ ,$$

$$\Gamma = Ext_R^1(-, V) : R\text{-}Mod \longrightarrow Mod\text{-}B \text{ and}$$

$$\Gamma = Ext_B^1(-, V) : Mod\text{-}B \longrightarrow R\text{-}Mod$$

with the natural transformations

$$\delta : Id_{R\text{-}Mod} \longrightarrow \Delta^2 \text{ and } \delta : Id_{Mod\text{-}B} \longrightarrow \Delta^2$$

given by evaluation . Now we have

Proposition 5.1.1 [CbF1 , proposition 1.1] *Let ${}_RV$ be a tilting module and $B = End({}_RV)$. Then ${}_RV_B$ is a faithfully balanced bimodule and V_B is a tilting module .*

Proof :
We have the exact sequences

$$(ex1): \quad 0 \to P_1 \to P_0 \to {}_RV \to 0$$

with P_i finitely generated and projective and

$$(ex2): \quad 0 \to_R R \to V_1 \to V_2 \to 0$$

with $V_i \in add({}_RV)$. Applying $\Delta = Hom_R(-,V)$ to $(ex1)$, since $\Gamma(V) = Ext^1_R(V,V) = 0$, we get the exact sequence

$$0 \to Hom_R(V,V) \to Hom_R(P_0,V) \to Hom_R(P_1,V) \to Ext^1_R(V,V) = 0$$

with $Hom_R(P_0,V)$, $Hom_R(P_1,V) \in add(V_B)$.

Applying $\Delta = Hom_R(-,V)$ to $(ex2)$, we have the exact sequence

$$(ex3): \quad 0 \to \Delta(V_2) \to \Delta(V_1) \to \Delta(R) \to \Gamma(T_2) = 0$$

. So , since $\Delta(R) \cong V_B$ and $\Delta(V_i)$ are finitely generated projective B-modules , we have the required projective resolution of V_B .

Applying $\Delta = Hom_B(-,V)$ to $(ex3)$, we get the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & {}_RR \arrow[r] \arrow[d, "\delta_R"'] & V_1 \arrow[r] \arrow[d, "\delta_{V_1}"] & V_2 \arrow[r] \arrow[d, "\delta_{V_2}"] & 0 & \\
0 \arrow[r] & \Delta^2({}_RR) \arrow[r] & \Delta^2(V_1) \arrow[r] & \Delta^2(V_2) \arrow[r] & \Gamma\Delta({}_RR) \arrow[r] & 0
\end{tikzcd}$$

where δ_{V_i} are isomorphisms since $V_i \in add({}_RV)$ and $\Gamma\Delta(V_1) = 0$ as $\Delta(V_1)$ is a projective B-module . And from this diagram , we have that $Ext^1_B(V,V) = \Gamma\Delta({}_RR) = 0$ and δ_R is an isomorphism , hence ${}_RV_B$ is faithfully balanced [AF , 20.15] . □

Let $\mathcal{T} = KerH'$, $\mathcal{F} = KerH$, $\mathcal{S} = KerT$ and $\mathcal{E} = KerT'$.
Before we prove the Tilting Theorem , we need two lemmas .

Lemma 5.1.2 [CbF , lemma 1.3] *Let ${}_RV$ be a tilting module and $B = End({}_RV)$. Then*

(1) If $M_3 \xrightarrow{h} M_2 \xrightarrow{g} M_1 \xrightarrow{f} M_0$ is an exact sequence in R-Mod with all $M_i \in \mathcal{T} = KerH' = Gen({}_RV)$, then the induced sequence

$$H(M_2) \longrightarrow H(M_1) \longrightarrow H(M_0)$$

is exact .

(2) If $M \in \mathcal{T} = KerH' = Gen({}_RV)$, then there is an exact sequence

$$... \longrightarrow V_n \longrightarrow V_{n-1} \longrightarrow ... \longrightarrow V_2 \longrightarrow V_1 \longrightarrow V_0 \longrightarrow M \longrightarrow 0$$

with $V_i \in Add({}_RV)$ for all $i \in \mathbb{N}$.

Proof :
(1) We have the exact sequences

$$0 \longrightarrow Imh \longrightarrow M_2 \longrightarrow Img \longrightarrow 0$$

$$0 \longrightarrow Img \longrightarrow M_1 \longrightarrow Imf \longrightarrow 0$$

$$0 \longrightarrow Imf \longrightarrow M_0 \longrightarrow Cokerf \longrightarrow 0$$

with Imh , Img and $Imf \in \mathcal{T} = Ker(H') = Ker(Ext^1_R(V, -)) = Gen({}_RV)$, here $\mathcal{T}$ is closed under epimorphic image . Then applying $H = Hom_R(V, -)$, we get the exact sequences

$$0 \longrightarrow H(Imh) \longrightarrow H(M_2) \longrightarrow H(Img) \longrightarrow H'(Imh) = 0 \quad ,$$

$$0 \longrightarrow H(Img) \longrightarrow H(M_1) \longrightarrow H(Imf) \longrightarrow H'(Img) = 0 \quad \text{and}$$

$$0 \longrightarrow H(Imf) \longrightarrow H(M_0) \longrightarrow H(Cokerf) \longrightarrow H'(Imf) = 0$$

. So the induced sequence $H(M_2) \longrightarrow H(M_1) \longrightarrow H(M_0)$ is exact .

(2) If $M \in \mathcal{T} = KerH' = Gen({}_RV)$ then for $X = H(M) = Hom_R(V, M)$, we have

$$\sum_{x \in X} x(V) = M \quad \text{and} \quad f \overset{def}{=} \bigoplus_{x \in X} x : V^{(X)} \longrightarrow M \quad \text{is epic}$$

i.e. we get the exact sequence

$$(ex0): \quad 0 \longrightarrow K_0 = Kerf \longrightarrow V_0 = V^{(X)} \xrightarrow{f} M \longrightarrow 0$$

. By construction , $H(ex0)$ is exact and so $K_0 \in Gen(_RV) = \mathcal{T} = KerH'$ and clearly $V_0 = V^{(X)} \in Gen(_RV)$.

Now start the construction again by K_0 instead of M , similarly , we get an exact sequence

$$(ex1): \quad 0 \to K_1 \to V_1 \to K_0 \to 0$$

with $V_1 \in Add(_RV)$ and $K_1 \in Gen(_RV) = \mathcal{T} = KerH'$. Repeating the process , we have the exact sequence

$$\ldots \to V_n \to V_{n-1} \to \ldots \to V_2 \to V_1 \to V_0 \to M \to 0$$

with $V_i \in Add(_RV)$ for all $i \in \mathbb{N}$. □

Lemma 5.1.3 *Let*

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & Ker(\gamma) \arrow[d, "\lambda"] & \\
 & M' \arrow[r, "\xi"] \arrow[d, "\alpha"] & M \arrow[r, "p"] \arrow[d, "\beta"] & M'' \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & N' \arrow[r, "j"] \arrow[d, "\mu"] & N \arrow[r, "\varphi"] & N'' & \\
 & Coker(\alpha) \arrow[d] & & & \\
 & 0 & & &
\end{tikzcd}$$

be a commutative diagram with exact rows and columns . By snake lemma , we have an exact sequence

$$Ker(\alpha) \to Ker(\beta) \to Ker(\gamma) \xrightarrow{\partial} Coker(\alpha) \to Coker(\beta) \to Coker(\gamma)$$

where $\partial(z) \overset{def}{=} \mu j^{-1}\beta p^{-1}\lambda(z)$ *for every* $z \in Ker(\gamma)$ *and* $p^{-1}\lambda(z)$ *is an element in* M *such that* $p(p^{-1}\lambda(z)) = \lambda(z)$. *Suppose now that* β *is an isomorphism , then* ∂ *is also an isomorphism and*

$$p \circ \beta^{-1} \circ j = \lambda \circ \partial^{-1} \circ \mu : N' \longrightarrow M''$$

Proof :
Clear . □

Now we can give a general version of

The Tilting Theorem 5.1.4 [CbF1 , the Tilting Theorem 1.4] *Suppose* ${}_RV$ *is a tilting module in* R-Mod *and* $B = End({}_RV)$. *Let*

$$H = Hom_R(V,-) \ , \ H' = Ext^1_R(V,-) \ , \ T = (V \otimes_B -) \ , \ T' = Tor^B_1(V,-)$$

so that we have the pairs of functors

$$H : R\text{-}Mod \rightleftarrows B\text{-}Mod : T \quad and \quad H' : R\text{-}Mod \rightleftarrows B\text{-}Mod : T'$$

and let

$$\mathcal{T} = KerH' \ , \ \mathcal{F} = KerH \ , \ \mathcal{S} = KerT \quad and \quad \mathcal{E} = KerT'$$

. *Then*

(1) $TH' = 0_{R\text{-}Mod} = T'H$ *and* $HT' = 0_{B\text{-}Mod} = H'T$.

(2) *There are natural transformations*

$$\theta : H'T' \longrightarrow Id_{B\text{-}Mod} \quad and \quad \eta : Id_{R\text{-}Mod} \longrightarrow T'H'$$

that , together with the canonical natural transformations ρ *and* σ *associated with* T *and* H , *yield exact sequences*

$$0 \longrightarrow THM \xrightarrow{\rho_M} M \xrightarrow{\eta_M} T'H'M \longrightarrow 0$$

$$0 \longrightarrow H'T'N \xrightarrow{\theta_N} N \xrightarrow{\sigma_N} HTN \longrightarrow 0$$

for each $M \in R$-Mod *and* $N \in B$-Mod .

(3) *The restrictions*

$$H : \mathcal{T} \rightleftarrows \mathcal{E} : T \quad \text{and} \quad H' : \mathcal{F} \rightleftarrows \mathcal{S} : T'$$

define category equivalences .

(4) $(\mathcal{T}, \mathcal{F})$ *and* $(\mathcal{S}, \mathcal{E})$ *are torsion theories in* R-Mod *and* B-Mod *, respectively .*

Proof :
(1) Since V is a tilting module , we have the category equivalence

$$H : Gen(_RV) \rightleftarrows Cogen(Hom_R(V, Q)) : T$$

where $_RQ$ is an injective cogenerator and $KerH' = Gen(_RV) = ImT$ and $KerT' = Cogen(Hom_R(V, Q)) = ImH$. So $T'H = 0_{R\text{-}Mod}$ and $H'T = 0_{B\text{-}Mod}$.

Now if $M \in R$-Mod there is an exact sequence

$$0 \to M \to C \to D \to 0$$

with $C = E(M)$, $D = E(M)/M \in \mathcal{T} = KerH' = Gen(_RV)$. Applying H , we have the exact sequence

$$0 \to H(M) \to H(C) \to H(D) \to H'(M) \to H'(C) = 0$$

and so we get a commutative diagram with exact rows

$$\begin{array}{ccccccc} C & \longrightarrow & D & \longrightarrow & 0 & & \\ {\scriptstyle \rho_C}\uparrow & & \uparrow{\scriptstyle \rho_D} & & & & \\ TH(C) & \longrightarrow & TH(D) & \longrightarrow & TH'(M) & \longrightarrow & 0 \end{array}$$

where ρ_C and ρ_D are isomorphisms , hence $TH'M = 0$. Note that $Ker(T')$ contains all projective left B-modules and is closed under sudmodules . Thus , if $_BN \in B$-Mod and

$$0 \to K \to P \to N \to 0$$

is exact with P projective , applying T , we have the exact sequence

$$0 = T'(P) \to T'(N) \to T(K) \to T(P) \to T(N) \to 0$$

and then the commutative diagram with exact rows and isomorphic columns

$$\begin{array}{ccccccc} & & 0 & \longrightarrow & K & \longrightarrow & P \\ & & & & \sigma_K \big\downarrow & & \big\downarrow \sigma_P \\ 0 & \longrightarrow & HT'(N) & \longrightarrow & HT(K) & \longrightarrow & HT(P) \end{array}$$

shows that $HT'N = 0$.
(2) (i) By (1) , we have the exact sequences

$$0 \to M \xrightarrow{\alpha} C \xrightarrow{\beta} D \to 0$$

with $C = E(M)$, $D = E(M)/M \in \mathcal{T} = KerH' = Gen(_RV)$ and

$$0 \to H(M) \to H(C) \xrightarrow{H(\beta)} H(D) \to H'(M) \to 0$$

. And from this , we obtain two short exact sequences

$$(ex1): \quad 0 \to H(M) \to H(C) \xrightarrow{\pi} L \to 0$$

$$(ex2): \quad 0 \to L \xrightarrow{j} H(D) \to H'(M) \to 0$$

where $j \circ \pi = H(\beta)$. Note that V_B is a tilting module , so flat dimension of $V_B \leq 1$. Applying T to $(ex2)$, we get

$$\ldots \to Tor_2^B(V, H'(M)) = 0 \to Tor_1^B(V, L) \to Tor_1^B(V, H(D)) = T'HD = 0$$

where $Tor_2^B(V, H'(M)) = 0$ as flat dimension of $V_B \leq 1$. So $T'(L) = Tor_1^B(V, L) = 0$. Hence , by applying T to $(ex1)$,

$$0 = T'L \to THM \to THC \xrightarrow{T(\pi)} TL \to 0$$

is exact . Applying T to $(ex2)$, as $T'HD = 0 = TH'M$, we have the exact sequence

$$0 = T'HD \to T'H'M \xrightarrow{\lambda} TL \xrightarrow{T(j)} THD \overset{\rho_D}{\cong} D \to TH'M = 0$$

where $\lambda =$ canonical morphism . Hence we get the exact sequence

$$0 \to T'H'M \xrightarrow{\lambda} TL \xrightarrow{\rho_D \circ T(j)} D \to 0$$

and the commutative diagram with exact rows and columns

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & & T'H'M \arrow[d, "\lambda"] & \\
0 \arrow[r] & THM \arrow[r, "TH(\alpha)"] \arrow[d, "\rho_M"'] & THC \arrow[r, "T(\pi)"] \arrow[d, "\rho_C", "\cong"'] & TL \arrow[r] \arrow[d, "\rho_D \circ T(j)"] & 0 \\
0 \arrow[r] & M \arrow[r, "\alpha"] \arrow[d, "\mu"'] & C \arrow[r, "\beta"] & D \arrow[r] \arrow[d] & 0 \\
 & M/Im(\rho_M) \arrow[d] & & 0 & \\
 & 0 & & &
\end{tikzcd}
$$

and from this diagram , ρ_M is monic [AF , lemma 3.14] . Also by the snake lemma , we have a connecting morphism $\partial : T'H'M \to M/Im(\rho_M)$ which is an isomorphism because ρ_C is . Then we have

$$M \xrightarrow{\mu} M/Im(\rho_M) \overset{\partial^{-1}}{\cong} T'H'M$$

where $\mu =$ natural epimorphism . Thus there is an epimorphism

$$\eta_M \overset{def}{=} \partial^{-1} \circ \mu : M \to T'H'M$$

with kernel $Im(\rho_M)$ i.e. we have the exact sequence

$$0 \to THM \xrightarrow{\rho_M} M \xrightarrow{\eta_M} T'H'M \to 0$$

. And by lemma 5.1.3 ,

$$\lambda \circ \eta_M = T(\pi) \circ \rho_C^{-1} \circ \alpha : M \to TL$$

Now we need to show that η is natural . To see this , let ${}_RM'$ be a R-module and $f : M \longrightarrow M'$ be a R-morphism . For ${}_RM'$, similarly , we have an exact sequence

$$0 \rightarrow M' \xrightarrow{\alpha'} C' \xrightarrow{\beta'} D' \rightarrow 0$$

with C' injective and a commutative diagram with exact rows

$$(G1) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{\alpha} & C & \xrightarrow{\beta} & D & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \\ 0 & \longrightarrow & M' & \xrightarrow{\alpha'} & C' & \xrightarrow{\beta'} & D' & \longrightarrow & 0 \end{array}$$

. Now applying H to $(G1)$ we get the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & HM & \longrightarrow & HC & \xrightarrow{H(\beta)} & HD & \longrightarrow & H'M & \longrightarrow & H'C = 0 \\ & & \downarrow{\scriptstyle Hf} & & \downarrow{\scriptstyle Hf_1} & & \downarrow{\scriptstyle Hf_2} & & \downarrow{\scriptstyle H'f} & & \\ 0 & \longrightarrow & HM' & \longrightarrow & HC' & \xrightarrow{H(\beta')} & HD' & \longrightarrow & H'M' & \longrightarrow & H'C' = 0 \end{array}$$

. Letting $L' = ImH(\beta')$ and factoring $H(\beta') = j' \circ \pi'$ as above we obtain two commutative diagrams with exact rows

$$(G2) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & HM & \longrightarrow & HC & \xrightarrow{\pi} & L & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Hf} & & \downarrow{\scriptstyle Hf_1} & & \downarrow{\scriptstyle \nu} & & \\ 0 & \longrightarrow & HM' & \longrightarrow & HC' & \xrightarrow{\pi'} & L' & \longrightarrow & 0 \end{array}$$

where , note that $Ker\pi \subseteq Ker(\pi' \circ Hf_1)$, ν is defined so that this diagram is commutative and

$$(G3) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{j} & HD & \longrightarrow & H'M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle Hf_2} & & \downarrow{\scriptstyle H'f} & & \\ 0 & \longrightarrow & L' & \xrightarrow{j'} & HD' & \longrightarrow & H'M' & \longrightarrow & 0 \end{array}$$

here the diagram $(G3)$ is commutative because

$$\begin{aligned}
j' \circ \nu \circ \pi &= j' \circ \pi' \circ Hf_1 \\
&= H(\beta') \circ H(f_1) \\
&= H(\beta' \circ f_1) \\
&= H(f_2 \circ \beta) \\
&= H(f_2) \circ H(\beta) \\
&= H(f_2) \circ j \circ \pi
\end{aligned}$$

and π is epic .

Note that , for ${}_RM'$, similarly we also have the cannonical morphism $\lambda' : T'H'M' \longrightarrow TL'$ which is monic and

$$\lambda' \circ \eta_{M'} = T(\pi') \circ \rho_{C'}^{-1} \circ \alpha' : M' \longrightarrow TL'$$

. Now we prove the diagram

$$\begin{array}{ccc}
M & \xrightarrow{\eta_M} & T'H'M \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle T'H'f} \\
M' & \xrightarrow{\eta_{M'}} & T'H'M'
\end{array}$$

is commutative and it suffices to show

$$\lambda' \circ \eta_{M'} \circ f = \lambda' \circ T'H'f \circ \eta_M$$

since λ' is monic .

$$\begin{aligned}
\lambda' \circ \eta_{M'} \circ f &= T(\pi') \circ \rho_{C'}^{-1} \circ \alpha' \circ f \\
&= T(\pi') \circ \rho_{C'}^{-1} \circ f_1 \circ \alpha \\
&= T(\pi') \circ THf_1 \circ \rho_C^{-1} \circ \alpha \\
&= T(\pi' \circ Hf_1) \circ \rho_C^{-1} \circ \alpha \\
&= T(\nu \circ \pi) \circ \rho_C^{-1} \circ \alpha \\
&= T(\nu) \circ T(\pi) \circ \rho_C^{-1} \circ \alpha \\
&= T(\nu) \circ \lambda \circ \eta_M \\
&= \lambda' \circ T'H'f \circ \eta_M
\end{aligned}$$

here the last equality holds because if we apply T to diagram $(G3)$, we can see that $T(\nu) \circ \lambda = \lambda' \circ T'H'f$.

(ii) Similarly , for ${}_BN \in B\text{-}Mod$ and let

$$0 \longrightarrow K \longrightarrow P \longrightarrow N \longrightarrow 0$$

be exact with P projective . Applying T we have the exact sequence

$$0 = T'P \rightarrow T'N \rightarrow TK \rightarrow TP \rightarrow TN \rightarrow 0$$

. Letting $L = KerT(\pi)$ we obtain two exact sequences

$$0 \rightarrow T'N \rightarrow TK \rightarrow L \rightarrow 0$$

$$0 \rightarrow L \rightarrow TP \rightarrow TN \rightarrow 0$$

. Note that $TK \in \mathcal{T} = KerH' = Gen({}_RV)$, so $L \in \mathcal{T}$. Applying H we get two exact sequences

$$0 = HT'N \rightarrow HTK \rightarrow HL \rightarrow H'T'N \rightarrow H'TK = 0$$

$$0 \rightarrow HL \rightarrow HTP \rightarrow HTN \rightarrow H'L = 0$$

and also the commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & P \arrow[r] \arrow[d, "\cong"'] & N \arrow[r] \arrow[d, "\sigma_N"] & 0 \\
0 \arrow[r] & HL \arrow[r] \arrow[d] & HTP \arrow[r] & HTN \arrow[r] & 0 \\
 & H'T'N \arrow[d] & & & \\
 & 0 & & &
\end{tikzcd}$$

. From this it follows that σ_N is epic and there is a monomorphism $\theta_N : H'T'N \rightarrow N$ with image $Ker(\sigma_N)$ i.e. the sequence

$$0 \rightarrow H'T'N \xrightarrow{\theta_N} N \xrightarrow{\sigma_N} HTN \rightarrow 0$$

is exact . It also follows that θ defines a natural transformation .

(3) Clear .

(4) ($\mathcal{S} = KerT, \mathcal{E} = KerT'$) is a torsion pair in B-Mod by proposition 3.2.2(3) . Moreover since ${}_RV$ is a tilting module , by theorem 3.4.7(3) ,

$$Gen({}_RV) = \{M_R \mid Ext^1_R(V, M) = 0\} = KerH' = \mathcal{T}$$

which is closed under extensions . Hence $Gen({}_RV)$ is a torsion class and the corresponding torsion-free class is

$$Ker(H) = \{L_R \mid Hom_R(V, L) = 0\}$$

since $Hom_R(N, {}_RL) = 0$ for all $N \in Gen({}_RV)$ iff $Hom_R(V, L) = 0$. So $(Gen({}_RV), \mathcal{F})$ is a torsion pair in R-Mod . □

Here it is worthy to point out that the following converses of the Tilting Theorem hold . Therefore the Tilting Theorem actually characterizes the tilting modules .

Theorem 5.1.5 [CbF2 , theorem 1.1] *Suppose ${}_RV$ is a module in R-Mod and let $B = End({}_RV)$. With notation as in the Tilting Theorem , assume*

(1) $TH' = 0_{R\text{-}Mod}$.

(2) *There are natural transformations θ and η such that for each $M \in R$-Mod and $N \in B$-Mod the sequences*

$$THM \xrightarrow{\rho_M} M \xrightarrow{\eta_M} T'H'M \to 0 \quad \text{and}$$

$$H'T'N \xrightarrow{\theta_N} N \xrightarrow{\sigma_N} HTN \to 0$$

are exact .

Then ${}_RV$ is a tilting module .

Proof :
It suffices to show that ${}_RV$ is selfsmall and $Gen({}_RV) = KerH' = \mathcal{T}$.
(i) By hypothesis (2) , $\sigma_{B^{(X)}}$ is an isomorphism , so we have canonical isomorphisms

$$H(V^{(X)}) \cong H(T(B)^{(X)}) \cong HT(B^{(X)}) \cong B^{(X)} = H(V)^{(X)}$$

for any set X .
(ii) Let $M \in R\text{-}Mod$, if $M \in KerH'$ then $T'H'M = 0$ and so ρ_M is epic i.e. $M \in Gen({}_RV)$.

On the other hand , if $M \in Gen({}_RV)$ then $T'H'M = 0$ and from the exact sequence

$$H'T'H'M = 0 \longrightarrow H'M \longrightarrow HTH'M = 0 \longrightarrow 0$$

we have $H'M = 0$. □

Theorem 5.1.6 [CbF2 , theorem 1.2] *Suppose ${}_RV$ is a module in $R\text{-}Mod$ and let $B = End({}_RV)$. With notation as in the Tilting Theorem , assume*

(1) $(\mathcal{T}, \mathcal{F}) = (KerH', KerH)$ *and* $(\mathcal{S}, \mathcal{E}) = (KerT, kerT')$ *are torsion theories .*

(2) *The restrictions* $H : \mathcal{T} \rightleftarrows \mathcal{E} : T$ *define a category equivalence .*

Then ${}_RV$ is a tilting module .

Proof :
It is clear by theorem 3.4.9 . □

5.2 Tilting Torsion Theories

Let ${}_RV$ be a tilting module and $B = End({}_RV)$. As before , we define the following functors :

$$H_{_RV} = Hom_R(V, -) : R\text{-}Mod \longrightarrow B\text{-}Mod$$

$$H'_{_RV} = Ext^1_R(V, -) : R\text{-}Mod \longrightarrow B\text{-}Mod$$

$$T_{V_B} = (V \otimes_B -) : B\text{-}Mod \longrightarrow R\text{-}Mod$$

$$T'_{V_B} = Tor^B_1(V, -) : B\text{-}Mod \longrightarrow R\text{-}Mod$$

and if we let $\mathcal{T}_{_RV} = Ker(H'_{_RV})$, $\mathcal{F}_{_RV} = Ker(H_{_RV})$, $\mathcal{S}_{V_B} = Ker(T_{V_B})$ and $\mathcal{E}_{V_B} = Ker(T'_{V_B})$ then by the Tilting Theorem , $(\mathcal{T}_{_RV}, \mathcal{F}_{_RV})$ and $(\mathcal{S}_{V_B}, \mathcal{E}_{V_B})$ are torsion theories and the restrictions

$$H_{_RV} : \mathcal{T}_{_RV} \rightleftarrows \mathcal{E}_{V_B} : T_{V_B} \quad \text{and} \quad H'_{_RV} : \mathcal{F}_{_RV} \rightleftarrows \mathcal{S}_{V_B} : T'_{V_B}$$

define category equivalences . Moreover since V_B is also a tilting module , we have the corresponding functors :

$$H_{V_B} = Hom_B(V, -) : Mod\text{-}B \longrightarrow Mod\text{-}R$$

$$H'_{V_B} = Ext^1_B(V, -) : Mod\text{-}B \longrightarrow Mod\text{-}R$$

$$T_{_RV} = (- \otimes_R V) : Mod\text{-}R \longrightarrow Mod\text{-}B$$

$$T'_{_RV} = Tor^R_1(-, V) : Mod\text{-}R \longrightarrow Mod\text{-}B$$

and torsion theories $(\mathcal{T}_{V_B}, \mathcal{F}_{V_B})$ and $(\mathcal{S}_{_RV}, \mathcal{E}_{_RV})$ where $\mathcal{T}_{V_B}$, $\mathcal{F}_{V_B}$, $\mathcal{S}_{_RV}$ and $\mathcal{E}_{_RV}$ denote the kernels of the functors H'_{V_B} , H_{V_B} , $T_{_RV}$ and $T'_{_RV}$ respectively . And the restrictions

$$H_{V_B} : \mathcal{T}_{V_B} \rightleftarrows \mathcal{E}_{_RV} : T_{_RV} \quad \text{and} \quad H'_{V_B} : \mathcal{F}_{V_B} \rightleftarrows \mathcal{S}_{_RV} : T'_{_RV}$$

define category equivalences .

Because of these , the problem of when a torsion theory is induced by a tilting module has a special interest . Torsion theories induced by tilting modules are called tilting torsion theories . This problem has been extensively studied by many authors . Here we only consider the problem of when two tilting modules , $_RV$ and $_RV'$, define the same torsion theories . Some uesful results for this problem were obtained by P.A. Guil Asensio and F. Guil Asensio [GG , 1992] .

Let R be a ring , $_RV$ and $_RV'$, two tilting modules , $B = End(_RV)$ and $B' = End(_RV')$. Then for the functors and torsion theories associated with V and V' defined above , we first have the

Lemma 5.2.1 [GG , lemma 2.1] *The following conditions are equivalent :*

(1) $\mathcal{T}_{_RV} \subseteq \mathcal{T}_{_RV'}$.

(2) $\mathcal{F}_{_RV'} \subseteq \mathcal{F}_{_RV}$.

(3) $\mathcal{E}_{_RV} \subseteq \mathcal{E}_{_RV'}$.

(4) $\mathcal{S}_{_RV'} \subseteq \mathcal{S}_{_RV}$.

(5) $Ext^1_R(V', V) = 0$.

Proof :
(1) $\Leftrightarrow$ (2) and (3) $\Leftrightarrow$ (4) are clear because $(\mathcal{T}_{RV}, \mathcal{F}_{RV})$ and $(\mathcal{T}_{RV'}, \mathcal{F}_{RV'})$ are torsion theories .
(1) $\Rightarrow$ (5) If $\mathcal{T}_{RV} \subseteq \mathcal{T}_{RV'}$ then since ${}_RV \in \mathcal{T}_{RV}$ we have ${}_RV \in \mathcal{T}_{RV'}$ i.e. $Ext^1_R(V', V) = 0$.
(5) $\Rightarrow$ (1) If $Ext^1_R(V', V) = 0$, then ${}_RV \in \mathcal{T}_{RV'} = KerH'_{RV'} = Gen({}_RV')$, so

$$\mathcal{T}_{RV} = KerH'_{RV} = Gen({}_RV) \subseteq \mathcal{T}_{RV'} \ .$$

(3) $\Rightarrow$ (5) Suppose that $\mathcal{E}_{RV} \subseteq \mathcal{E}_{RV'}$ and let E_B be an injective cogenerator of $Mod\text{-}B$. Then $Ext^1_B(V, E) = 0$. By Tilting Theorem , $Im(H_{V_B}) \subseteq Ker(T'_{RV})$, so

$$Hom_B(V, E) \in \mathcal{E}_{RV} \subseteq \mathcal{E}_{RV'} \ .$$

And , as ${}_RV'$ is a tilting module , there is an exact sequence

$$(ex1): \quad 0 \longrightarrow P_0 \longrightarrow P_1 \longrightarrow_R V' \longrightarrow 0$$

with P_0 and P_1 finitely generated and projective left R-modules . Applying the functor $(Hom_B(V, E) \otimes_R -)$ to $(ex1)$, we get the exact sequence

$$0 \longrightarrow Tor_1^R(Hom_B(V, E), V') \longrightarrow Hom_B(V, E) \otimes_R P_0$$
$$\longrightarrow Hom_B(V, E) \otimes_R P_1 \longrightarrow Hom_B(V, E) \otimes_R V' \longrightarrow 0$$

where $Tor_1^R(Hom_B(V, E), P_1) = 0$. On the other hand , if we apply the functor $Hom_R(-, {}_RV)$ to $(ex1)$, we get

$$0 \longrightarrow Hom_R(V', V) \longrightarrow Hom_R(P_1, V) \longrightarrow Hom_R(P_0, V)$$
$$\longrightarrow Ext^1_R(V', V) \longrightarrow Ext^1_R(P_1, V) = 0$$

and then applying the exact functor $Hom_B(, E_B)$, here E_B is injective , we have the exact sequence

$$0 \longrightarrow Hom_B(Ext^1_R(V', V), E) \longrightarrow Hom_B(Hom_R(P_0, V), E)$$
$$\longrightarrow Hom_B(Hom_R(P_1, V), E) \longrightarrow Hom_B(Hom_R(V', V), E) \longrightarrow 0$$

. Now we obtain a commutative diagram with exact rows :

$$\begin{tikzcd}
0 \arrow[r] & Tor_1^R(Hom_B(V, E), V') \arrow[r] & Hom_B(V, E) \otimes_R P_0 \arrow[d, "\nu"', "\cong"] \\
0 \arrow[r] & Hom_B(Ext^1_R(V', V), E) \arrow[r] & Hom_B(Hom_R(P_0, V), E)
\end{tikzcd}$$

$$\begin{array}{ccccc}
\longrightarrow & Hom_B(V,E)\otimes_R P_1 & \longrightarrow & Hom_B(V,E)\otimes_R V' & \longrightarrow 0 \\
& \nu\downarrow\cong & & \nu\downarrow & \\
\longrightarrow & Hom_B(Hom_R(P_1,V),E) & \longrightarrow & Hom_B(Hom_R(V',V),E) & \longrightarrow 0
\end{array}$$

where ν is defined as in [AF , 20.11] and note that here P_0 and P_1 are finitely generated and projective . Now

$$Hom_B(V,E) \in \mathcal{E}_{RV} \subseteq \mathcal{E}_{RV'}$$

and so $Tor_1^R(Hom_B(V,E),V') = 0$. Hence $Hom_B(Ext_R^1(V',V),E) = 0$. Since E_B is a cogenerator , $Ext_R^1(V',V) = 0$.

(5) $\Rightarrow$ (3) Suppose now that $Ext_R^1(V',V) = 0$ and for any $M_R \in \mathcal{E}_{RV}$, since by Tilting Theorem

$$H_{V_B} : \mathcal{T}_{V_B} \rightleftarrows \mathcal{E}_{RV} : T_{RV}$$

defines an equivalence , there is an N_B such that $Hom_B(V,N) = M$. An argument similar to (3) $\Rightarrow$ (5) give us a commutative diagram with exact rows :

$$\begin{array}{ccccc}
0 & \longrightarrow & Tor_1^R(Hom_B(V,N),V') & \longrightarrow & Hom_B(V,N)\otimes_R P_0 \\
& & & & \cong\downarrow \\
& & 0 & \longrightarrow & Hom_B(Hom_R(P_0,V),N)
\end{array}$$

$$\begin{array}{cc}
\longrightarrow & Hom_B(V,N)\otimes_R P_1 \\
& \cong\downarrow \\
\longrightarrow & Hom_B(Hom_R(P_1,V),N)
\end{array}$$

and hence $Tor_1^R(Hom_B(V,N),V') = 0$ i.e. $M = Hom_B(V,N) \in \mathcal{E}_{RV'}$. $\square$

Now we can characterize when two tilting modules V and V' induce the same torsion theories .

Corollary 5.2.2 [GG , corollary 2.2] *The following conditions are equivalent* :

(1) $\mathcal{T}_{_RV} = \mathcal{T}_{_RV'}$.

(2) $\mathcal{F}_{_RV'} = \mathcal{F}_{_RV}$.

(3) $\mathcal{E}_{_RV} = \mathcal{E}_{_RV'}$.

(4) $\mathcal{S}_{_RV'} = \mathcal{S}_{_RV}$.

(5) $Ext^1_R(V', V) = 0 = Ext^1_R(V, V')$.

and in particular , $(\mathcal{T}_{_RV}, \mathcal{F}_{_RV}) = (\mathcal{T}_{_RV'}, \mathcal{F}_{_RV'})$ *if and only if* $(\mathcal{S}_{_RV}, \mathcal{E}_{_RV}) = (\mathcal{S}_{_RV'}, \mathcal{E}_{_RV'})$.

Proposition 5.2.3 [GG , proposition 2.3] *Let* ${}_RV$ *and* ${}_RV'$ *be two tilting modules and let* $B = End({}_RV)$, $B' = End({}_RV')$. *Then the following conditions are equivalent :*

(1) $V' \in add({}_RV)$.

(2) $Hom_R(V', V)$ *is a progenerator in* $Mod\text{-}B$ *and* $Ext^1_R(V', V) = 0$.

Proof :
(1) $\Rightarrow$ (2) Suppose that $V' \in add({}_RV)$. Then , since $Ext^1_R(V, V) = 0$, we have $Ext^1_R(V', V) = Ext^1_R(V, V') = 0$. Also we have a splitting exact sequence

$$(ex): \quad 0 \longrightarrow K \longrightarrow V^n \longrightarrow_R V' \longrightarrow 0$$

in $Mod\text{-}R$ and here $n \in \mathbb{N}$. Apply $Hom_R(-, V)$ to (ex) , we get a splitting exact sequence

$$0 \longrightarrow Hom_R(V', V) \longrightarrow B^n \longrightarrow Hom_R(K, V) \longrightarrow 0$$

in $Mod\text{-}B$ and so $Hom_R(V', V)$ is a finitely generated projective right B-module .

We need to prove that $Hom_R(V', V)$ is a generator in$Mod\text{-}B$. And it suffices to prove that for $M \in Mod\text{-}B$ with $Hom_B(Hom_R(V', V), M) = 0$ we have that $M = 0$. First we have an exact sequence

$$0 \longrightarrow R \longrightarrow V_1' \longrightarrow V_2' \longrightarrow 0$$

in R-Mod with V_1' , $V_2' \in add({}_RV')$. Applying $Hom_R(-,V)$, we have

$$0 \longrightarrow Hom_R(V_2',V) \longrightarrow Hom_R(V_1',V) \longrightarrow Hom_R(R,V) \longrightarrow Ext_R^1(V_2',V) = 0$$

where $Hom_R(R,V) \cong V$ and then applying $Hom_B(-,M)$, note that

$$Hom_B(Hom_R(V',V),M) = 0$$

and $Hom_R(V_1',V)$ is projective B-module , we have the exact sequence

$$0 \longrightarrow Hom_B(V,M) \longrightarrow Hom_B(Hom_R(V_1',V),M) = 0$$

$$\longrightarrow Hom_B(Hom_R(V_2',V),M) = 0$$

$$\longrightarrow Ext_B^1(V,M) \longrightarrow Ext_B^1(Hom_R(V_1',V),M) = 0$$

and so $Hom_B(V,M) = 0 = Ext_B^1(V,M)$, but then $M = 0$ as V_B is a tilting module .

(2) $\Rightarrow$ (1) If $Hom_R(V',V)$ is a progenerator in Mod-B and $Ext_R^1(V',V) = 0$, then $Hom_B(Hom_R(V',V),V) \in add({}_RV)$ because we have a splitting exact sequence in Mod-B :

$$0 \longrightarrow Hom_R(V',V) \longrightarrow B^n \longrightarrow L_B \longrightarrow 0$$

for some $n \in \mathbb{N}$, applyiny $Hom_B(-,V)$, we get the splitting exact sequence in R-Mod :

$$0 \longrightarrow Hom_B(L,V) \longrightarrow Hom_B(B^n,V) \longrightarrow Hom_B(Hom_R(V',V),V) \longrightarrow 0$$

where $Hom_B(B^n,V) \cong V^n$.

On the other hand , there is an exact sequence in R-Mod :

$$0 \longrightarrow P_0 \longrightarrow P_1 \longrightarrow_R V' \longrightarrow 0$$

with P_0 , P_1 finitely generated projective R-modules . Applying $Hom_R(-,V)$, by hypothesis $Ext_R^1(V',V) = 0$, we get

$$0 \longrightarrow Hom_R(V',V) \longrightarrow Hom_R(P_1,V) \longrightarrow Hom_R(P_0,V) \longrightarrow 0$$

and then applying $Hom_B(-,V)$, using that $Ext^1_R(V,V) = 0$, we get the commutative diagram with exact rows :

$$\begin{array}{ccccc} 0 \longrightarrow & P_0 & \longrightarrow & P_1 & \\ & e_{P_0}\Big\downarrow\cong & & e_{P_1}\Big\downarrow\cong & \\ 0 \longrightarrow & Hom_B(Hom_R(P_0,V),V) & \longrightarrow & Hom_B(Hom_R(P_1,V),V) & \\ \longrightarrow & {}_RV' & \longrightarrow 0 & & \\ & e_{V'}\Big\downarrow & & & \\ \longrightarrow & Hom_B(Hom_R(V',V),V) & \longrightarrow 0 & & \end{array}$$

where $Ext^1_B(Hom_R(P_0,V),V) = 0$ because $Ext^1_B(Hom_R(R^n,V),V) = 0$, note that $Ext^1_R(V,V) = 0$. Moreover the evaluation maps e_{P_0} and e_{P_1} are isomorphisms because , as ${}_RV_B$ is faithfully balanced (proposition 5.1.2) ,

$$e_{{}_RR} : R \longrightarrow Hom_B(Hom_R(R,V),V)$$

is an isomorphism (see also [AF , 20.16 and 20.13]) . Finally we obtain that

$$V' \cong Hom_B(Hom_R(V',V),V) \in add({}_RV) \ .$$

Note here that actually V' is V-reflexive . □

Proposition 5.2.4 [GG , proposition 2.4] *Let ${}_RV$ and ${}_RV'$ be two tilting modules and let $B = End({}_RV)$, $B' = End({}_RV')$. Then the following conditions are equivalent :*

(1) $Ext^1_R(V,V') = Ext^1_R(V',V) = 0$

(2) $V' \in add({}_RV)$.

(3) $V \in add({}_RV')$.

Proof :
Since $Ext^1_R(V,V) = Ext^1_R(V',V') = 0$, (2) ⇒ (1) and (3) ⇒ (1) are clear .

(1) ⇒ (2) Let ${}_RV'' = V \oplus V'$. Then V'' is a tilting module with $Ext^1_R(V,V'') =$

$Ext^1_R(V'',V) = 0$ and $V \in add({}_RV'')$ and so we can apply the last proposition . If we let $S = End({}_RV'')$, by the proposition 5.2.3 , then $Hom_R(V,V'')$ is a progenerator in $Mod\text{-}S$.

By Morita Theorem [AF , 22.2] , $Hom_S(Hom_R(V,V''),S_S)$ is a progenerator in $Mod\text{-}End(Hom_R(V,V'')_S)$. But , by [AF , 20.7] , we have the isomorphisms

$$Hom_S(Hom_R(V,V''),S_S) = Hom_S(Hom_R(V,V''),Hom_R(V'',V''))$$

$$\cong Hom_R(V'',Hom_S(Hom_R(V,V''),V'')) \cong Hom_R(V'',V)$$

where $V \cong Hom_S(Hom_R(V,V''),V'')$ i.e. V is V''-reflexive as in the proof of [proposition 5.2.3 (2) $\Rightarrow$ (1)] . Moreover , [AF , 20.7] , we have that

$$End(Hom_R(V,V'')_S) = Hom_S(Hom_R(V,V''),Hom_R(V,V''))$$

$$\cong Hom_R(V,Hom_S(Hom_R(V,V''),V'')) \cong Hom_R(T,T) = B$$

. Therefore $Hom_R(V'',V)$ is a progenerator in $Mod\text{-}B$. Now by proposition 5.2.3 , $V'' \in add({}_RV)$ and hence $V' \in add({}_RV)$.
(1) $\Rightarrow$ (3) Similar to (1) $\Rightarrow$ (2) . $\square$

Finally we obtain the main theorem :

Theorem 5.2.5 [GG , Theorem 2.5] *Let ${}_RV$ and ${}_RV'$ be two tilting modules , $B = End({}_RV)$ and $B' = End({}_RV')$. Then the following conditions are equivalent :*

(1) $(\mathcal{T}_{{}_RV},\mathcal{F}_{{}_RV}) = (\mathcal{T}_{{}_RV'},\mathcal{F}_{{}_RV'})$.

(2) $(\mathcal{S}_{{}_RV},\mathcal{E}_{{}_RV}) = (\mathcal{S}_{{}_RV'},\mathcal{E}_{{}_RV'})$.

(3) $Ext^1_R(V,V') = Ext^1_R(V',V) = 0$.

(4) $V' \in add({}_RV)$.

(5) $V \in add({}_RV')$.

(6) $Hom_R(V',V)$ *is a progenerator in* $Mod\text{-}B$ *and* $Ext^1_R(V',V) = 0$.

(7) $Hom_R(V,V')$ *is a progenerator in* $Mod\text{-}B'$ *and* $Ext^1_R(V,V') = 0$.

(8) *There is a category equivalence*

$$F : Mod\text{-}B \rightleftarrows Mod\text{-}B' : G$$

with $F(V_B) \cong V'_{B'}$.

Moreover , if condition (8) holds , there are natural isomorphisms

$$F \cong Hom_B(Hom_R(V', V), -) \quad \text{and} \quad G \cong Hom_{B'}(Hom_R(V, V'), -) \quad .$$

Proof :

(1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) By corollary 5.2.2 .
(3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) By proposition 5.2.4 .
(4) $\Leftrightarrow$ (6) By proposition 5.2.3 .
(5) $\Leftrightarrow$ (7) By proposition 5.2.3 .

(6) + (7) $\Rightarrow$ (8) Since $Hom_R(V', V)$ is a progenerator in $Mod\text{-}B$, by Morita's theorem [AF , 22.2] , if we let $F = Hom_B(Hom_R(V', V), -)$ and for $Q = F(B)$, let $G = Hom_B(Q, -)$ then

$$F : Mod\text{-}B \rightleftarrows Mod\text{-}B' : G$$

define an equivalence . And by proposition [5.2.3 (2) $\Rightarrow$ (1)] , we have that V' is V-reflexive i.e.

$$F(V) = Hom_B(Hom_R(V', V), V) \cong V'$$

. Also

$$\begin{aligned} Q = F(B) &= Hom_B(Hom_R(V', V), B) \\ &= Hom_B(Hom_R(V', V), Hom_R(V, V)) \\ &\cong Hom_R(V, Hom_B(Hom_R(V', V), V)) \\ &\cong Hom_R(V, V') \end{aligned}$$

, hence , $G \cong Hom_{B'}(Hom_R(V, V'), -)$.

(8) $\Rightarrow$ (6) Suppose that there is a category equivalence

$$F : Mod\text{-}B \rightleftarrows Mod\text{-}B' : G$$

with $F(V_B) \cong V'_{B'}$. Then by Morita's theorem [AF , 22.2] , set ${}_{B'}P_B = G(B')$ and $Q = Hom_B(P, B)$, we have

$$F \cong Hom_B(P, -) \cong (- \otimes {}_B Q) \quad \text{and}$$

$$G \cong Hom_{B'}(Q, -) \cong (- \otimes {}_{B'}P)$$

where ${}_{B'}P$, P_B , $Q_{B'}$ and ${}_BQ$ are progenerators . Hence

$$Q = Hom_B(P, B) = Hom_B(P, Hom_R(V, V)) \cong Hom_R(V, Hom_B(P, V))$$
$$\cong Hom_R(V, V') \text{ and}$$
$$P = G(B') \cong Hom_{B'}(Q, Hom_R(V', V')) \cong Hom_R(V', Hom_{B'}(Q, V'))$$
$$\cong Hom_R(V', V) \ .$$

Also $Ext^1_R(V', V) \cong Ext^1_R(FV, V) \cong Ext^1_R(V \otimes {}_BQ, V)$. But ${}_BQ$ is finitely generated and projective and $Ext^1_R(V, V) = 0$, so $Ext^1_R(V \otimes {}_BQ, V) = 0$, hence $Ext^1_R(V', V) = 0$.

The final assertion follows from the proof of (8) $\Rightarrow$ (6) . $\square$

5.3 Isomorphisms of Endomorphism Rings of Tilting Modules

We want to characterize the isomorphisms between the endomorphism rings of some classes of modules by equivalences between module subcategories . To be more precise , let R and S be two rings , ${}_R\mathcal{C}$ and ${}_S\mathcal{D}$ full subcategories of R-Mod and S-Mod respectively , ${}_RM \in {}_R\mathcal{C}$ and ${}_SN \in {}_S\mathcal{D}$ and let

$$\delta : End({}_RM) \to End({}_SN)$$

be a ring isomorphism . We say that δ is *induced by an equivalence* if there exists an equivalence

$$F_\delta : {}_R\mathcal{C} \to {}_S\mathcal{D}$$

and an isomorphism of left S-modules

$$\phi : F(M) \to N$$

such that we have the commutative diagram

$$\begin{array}{ccc} F(M) & \xrightarrow{F(f)} & F(M) \\ \phi\downarrow & & \downarrow\phi \\ N & \xrightarrow{\delta(f)} & N \end{array}$$

for every $f \in End(_RM)$.

Here we consider this problem for the classes of progenerators , quasi-progenerators and tilting modules . In 1984 , Bolla obtained the following

Theorem 5.3.1 [B , theorem 2.1] *Let $_RP$ and $_SQ$ be progenerators . Suppose that*

$$\theta : End(_RP) \longrightarrow End(_SQ)$$

is a ring isomorphism . Then there exists an equivalence

$$F_\theta : R\text{-}Mod \longrightarrow S\text{-}Mod$$

(unique up to natural isomorphism) such that $F_\theta(P) = Q$ and $F_\theta(f) = \theta(f)$ for all $f \in End(_RP)$.

Recently Bolla's theorem is generalized to the class of quasi-progenerators by Lok and Shum [LS , 1995] .

Theorem 5.3.2 [LS , theorem 2 , 1995] *Let $_RU$ and $_SV$ be two quasi-progenerators . let*

$$\theta : End(_RU) \longrightarrow End(_SV)$$

be a ring isomorphism . Then there exists a categorical equivalence

$$F_\theta : Gen(_RU) \longrightarrow Gen(_SV)$$

which is unique up to natural isomorphism such that $F_\theta(_RU) = {_SV}$ and $F_\theta(f) = \theta(f)$ for all $f \in End(_RU)$.

Note here that $Gen(_RU)$ and $Gen(_SV)$ are closed under submodules because $_RU$ and $_SV$ are quasi-progenerators .

On the other hand , Bolla's result is also extended to the class of tilting modules by P.A. Guil Asensio and F. Guil Asensio [GG , 1992] . However it should be noted that in general , the class of quasi-progenerators $\neq$ the class of tilting modules .

Let $_RV$ and $_SV'$ be two tilting modules , $B = End(_RV)$ and $B' = End(_RV')$. We first have the

Lemma 5.3.3 [GG , lemma 3.1] *let $_RV$ be a tilting module and suppose that there is an equivalence $F : R\text{-}Mod \rightleftarrows S\text{-}mod : G$. Then $F(V)$ is a tilting S-module .*

Proof :
By Morita Theorem , let $GS = {}_RP$ which is a progenerator in $R\text{-}Mod$ and we have $F \cong Hom_R(P, -)$ and $End({}_RP) \cong F({}_RP) \cong S$.

We first prove that if $_RP$ is finitely generated and projective then there is an exact sequence

$$0 \rightarrow_R P \rightarrow V_0 \rightarrow V_1 \rightarrow 0$$

with $V_i \in add({}_RV)$. Now suppose $_RR^n = P \oplus L$ for some $_RL$, then $Hom_R({}_RP, {}_RV_B) \in add(V_B)$. Since V_B is a tilting right B-module , V_B is finitely presented and $projdim(V_B) \leq 1$. By [W , 25.1 (iii)] and [R , proposition 5.1.20] , we have that $Hom_R({}_RP, {}_RV_B)$ is finitely presented and $projdim(V_B) \leq 1$. Therefore there is an exact sequence

$$0 \rightarrow K_1 \rightarrow K_0 \rightarrow Hom_R({}_RP, {}_RV_B) \rightarrow 0$$

with $K_i \in add(B_B)$. Now , applying $Hom_B(-, {}_RV_B)$, we get the exact sequence

$$0 \rightarrow Hom_B(Hom_R({}_RP, {}_RV_B), {}_RV_B) \rightarrow Hom_B(K_0, {}_RV_B)$$

$$\rightarrow Hom_B(K_1, {}_RV_B) \rightarrow Ext^1_B(Hom_R({}_RP, {}_RV_B), V_B) = 0$$

where $Hom_B(Hom_R({}_RP, {}_RV_B), {}_RV_B) \cong {}_RP$ [AF , 20.16] , $Hom_B(K_i, {}_RV_B) \in add({}_RV)$ and $Ext^1_B(Hom_R({}_RP, {}_RV_B), V_B) = 0$ because $Hom_R({}_RP, {}_RV_B) \in add(V_B)$.

Now for the progenerator $_RP$, by applying F , we get the exact sequence

$$0 \rightarrow F({}_RP) \cong S \rightarrow F(V_0) \rightarrow F(V_1) \rightarrow 0$$

with $F(V_0)$, $F(V_1) \in add(F({}_RV))$. □

Proposition 5.3.4 [GG , proposition 3.2] *Let $_RV$ and $_SV'$ be two tilting modules with $B = End({}_RV)$ and $B' = End({}_SV')$. Suppose that there is an equivalence*

$$F : B\text{-}Mod \rightleftarrows B'\text{-}mod : G$$

where , for some progenerator ${}_BP_{B'}$,

$$F = Hom_B({}_BP_{B'}, -) \quad \textit{and} \quad G = ({}_BP_{B'} \otimes {}_{B'}-) \quad .$$

If

$$Ext^1_{B'}(V \otimes P, V') = Ext^1_{B'}(V', V \otimes P) = 0$$

then there is a category equivalence

$$F' : R\text{-}Mod \rightleftarrows S\text{-}mod : G'$$

such that :

(i) $F'(V \otimes P) \cong V'$.

(ii) In the following diagram

$$\begin{tikzcd}
B\text{-}Mod \arrow[rr, shift left, "F"] \arrow[d, shift right, "(V \otimes {}_B-)"'] & & B'\text{-}Mod \arrow[ll, shift left, "G"] \arrow[d, shift right, "(V' \otimes {}_{B'}-)"'] \\
R\text{-}Mod \arrow[rr, shift left, "F'"] \arrow[u, shift right, "Hom_R(V,-)"'] & & S\text{-}Mod \arrow[ll, shift left, "G'"] \arrow[u, shift right, "Hom_S(V',-)"']
\end{tikzcd}$$

there are natural transformations

$$\eta : (V' \otimes {}_{B'}-) \circ F \circ Hom_R(V, -) \to F' \quad \textit{and}$$

$$\eta' : (V \otimes {}_B-) \circ G \circ Hom_S(V', -) \to G'$$

such that η_M *and* η'_N *are isomorphisms for any* $M \in \mathcal{T}_{RV}$, $N \in \mathcal{T}_{SV'}$.

Proof :
Since ${}_BP_{B'}$ is a progenerator , the functor $(- \otimes {}_BP_{B'})$ defines an equivalence between *Mod-B* and *mod-B'* . So , by lemma 6.1.3 , $V \otimes {}_BP_{B'}$ is a right tilting B'-module and by hypothesis ,

$$Ext^1_{B'}(V \otimes P, V') = Ext^1_{B'}(V', V \otimes P) = 0$$

and then , by theorem 5.2.5(8) , we have an equivalence

$$F' : R\text{-}Mod \rightleftarrows S\text{-}mod : G'$$

with (i) $F'(V \otimes P) \cong V'$ and $F' = Hom_R(Hom_{B'}(V', V \otimes P), -)$.

Now we show that there is a natural transformation

$$\eta : (V' \otimes_{B'} -) \circ F \circ Hom_R(V, -) \to F' \quad .$$

For $M \in R\text{-}Mod$, $F(Hom_R(V, M)) = Hom_B(P, Hom_R(V, M))$, by adjoint property [AF , 20.6] , there is a natural isomorphism

$$\alpha_M : F(Hom_R(V, M)) \to Hom_R(V \otimes_B P, M)$$

. Now using [AF , 20.11] , we get a natural homomorphism

$$\nu_M : V' \otimes_{B'} Hom_R(V \otimes_B P, M) \to Hom_R(Hom_{B'}(V', V \otimes P), M) = F'(M)$$

. Hence $\eta_M \overset{def}{=} \nu_M \circ (Id_{V'} \otimes \alpha_M)$ is a natural morphism .

Finally , for $M \in \mathcal{T}_{_RV} = \{_RM \mid Ext^1_R(V, M) = 0\}$, since $_BP$ is a finitely generated and projective B-module , we have $Ext^1_R(V \otimes_B P, M) = 0$.

Moreover , there is an exact sequence

$$(ex): \quad 0 \to K_0 \to K_1 \to V'_{B'} \to 0$$

with K_0 and K_1 finitely generated and projective B'-modules . Then we have the exact sequence

$$K_0 \otimes_{B'} Hom_R(V \otimes_B P, M) \to K_1 \otimes_{B'} Hom_R(V \otimes_B P, M)$$

$$\to V'_{B'} \otimes_{B'} Hom_R(V \otimes_B P, M) \to 0$$

. On the other hand , as $Ext^1_{B'}(V', V \otimes P) = 0$ and $Ext^1_R(V \otimes P, M) = 0$ and by [AF , 20.11] , we obtain the commutative diagram with exact rows

$$\begin{array}{ccccc}
 & & K_0 \otimes_{B'} Hom_R(V \otimes_B P, M) & \longrightarrow & K_1 \otimes_{B'} Hom_R(V \otimes_B P, M) \\
 & & \downarrow \cong & & \downarrow \cong \\
0 & \longrightarrow & Hom_R(Hom_{B'}(K_0, V \otimes P), M) & \longrightarrow & Hom_R(Hom_{B'}(K_1, V \otimes P), M)
\end{array}$$

$$\begin{array}{ccc}
\longrightarrow & V' \otimes_{B'} Hom_R(V \otimes_B P, M) & \\
 & \downarrow \nu_M & \\
\longrightarrow & Hom_R(Hom_{B'}(V', V \otimes P), M) & \longrightarrow 0
\end{array}$$

and from this diagram , ν_M is an isomorphism for $M \in \mathcal{T}_{_RV}$.

Similarly we can show that there is a natural morphism

$$\eta' : (V \otimes_B -) \circ G \circ Hom_S(V', -) \longrightarrow G'$$

such that η'_N is an isomorphism for any $N \in \mathcal{T}_{_SV'}$. □

Lemma 5.3.5 *Let $\beta : B \longrightarrow B'$ be a ring isomorphism and ${}_SM_B$ be a bimodule . If we treat , via β , B as a bimodule ${}_BB_{B'}$, B' as a bimodule ${}_BB'_{B'}$ and M as a bimodule ${}_SM_{B'}$ then*

$$\mu' : {}_SM_{B'} \longrightarrow {}_SM_B \otimes_B B'_{B'} \quad [m \mapsto m \otimes 1]$$

is a (S, B')-bimodule isomorphism .

Proof :
Now $\beta : {}_BB_{B'} \longrightarrow {}_BB'_{B'}$ is a (B, B')-bimodule isomorphism and so

$$Id_M \otimes \beta : {}_SM_B \otimes_B B_{B'} \longrightarrow {}_SM_B \otimes_B B'_{B'}$$

is a (S, B')-bimodule isomorphism . But , by [AF , 20.1(2)] ,

$$\mu : {}_SM_{B'} \longrightarrow {}_SM_B \otimes_B B_{B'} \quad [m \mapsto m \otimes 1]$$

is a (S, B')-bimodule isomorphism . Hence

$$\mu' \stackrel{def}{=} (Id_M \otimes \beta) \circ \mu : {}_SM_{B'} \longrightarrow {}_SM_B \otimes_B B'_{B'}$$

is a (S, B')-bimodule isomorphism . □

Suppose now that ${}_RV$ and ${}_SV'$ are two tilting modules with $B = End({}_RV)$ and $B' = End({}_SV')$ and that there is an equivalence

$$F : R\text{-}Mod \rightleftarrows S\text{-}Mod : G$$

with $F(V) = V'$. Let ${}_RP_S$ be a progenerator ($End({}_RP) = S$) and

$$\zeta : F \cong Hom_R(P, -)$$

be a natural isomorphism . By [AF , lemma 20.3] , we have the ring isomorphism

$$F : B = End(_RV) \longrightarrow B' = End(_SV')$$

and so , via F , we can treat $_SM_B = {}_SHom_R(_RP_S, {}_RV_B)_B$ as a (S, B')-bimodule $_SHom_R(_RP_S, {}_RV_B)_{B'}$. By the above lemma , we have a (S, B')-bimodule isomorphism

$$F^* : {}_SHom_R(_RP_S, {}_RV_B)_{B'} \longrightarrow Hom_R(_RP_S, {}_RV_B)_B \otimes_B B'_{B'}$$

defined by $F^*(f) = f \otimes 1$ for every $f \in Hom_R(P, V)$.

Proposition 5.3.6 [GG , lemma 3.3 and proposition 3.4] *With the above assumptions , the composition morphism*

$$F^* \circ \zeta_V : {}_SF(V)_{B'} \longrightarrow {}_SHom_R(_RP_S, {}_RV_B)_B \otimes_B B'_{B'}$$

is an isomorphism of (S, B')*-bimodules .*

Moreover

$$Ext^1_{B'}(V', V \otimes_B B') = Ext^1_{B'}(V \otimes_B B', V') = 0 \quad .$$

Proof :
Since $_RV_B$ is a bimodule and , via the ring isomorphism F , $F(V)$ has a bimodule structure $_SF(V)_B$. So by [AF , lemma 20.4(2)] ,

$$\zeta_V : {}_SF(V)_B \longrightarrow {}_SHom_R(_RP_S, {}_RV_B)_B$$

is an isomorphism of (S, B)-bimodules . But now

$$\zeta_V : {}_SF(V)_{B'} \longrightarrow {}_SHom_R(_RP_S, {}_RV_B)_{B'}$$

is also an isomorphism of (S, B')-bimodules and we obtain that $F^* \circ \zeta_V$ is an isomorphism of (S, B')-bimodules .

Since $_RP$ is a progenerator , $_RR$ is a direct summand of $_RP^n$ for some n . Thus $V \otimes_B B' \cong Hom_R(_RR, V) \otimes_B B'$ is a direct summand of

$$Hom_R(_RP^n, V) \otimes_B B' \cong (Hom_R(_RP, V) \otimes_B B')^n$$

as right B'-modules .

But now $Hom({}_RP,V)\otimes_B B' \cong F(V) = V'$ as right B'-modules and $Ext^1_{B'}(V',V')=0$ so we obtain that

$$Ext^1_{B'}(V',V\otimes_B B') = Ext^1_{B'}(V\otimes_B B',V') = 0 \ . \ \square$$

Finally we can characterizes an isomorphism between the endomorphism rings of two tilting modules by category equivalence .

Theorem 5.3.7 [GG , theorem 3,5] *Let ${}_RV$ and ${}_SV'$ be two tilting modules with $B = End({}_RV)$ and $B' = End({}_SV')$. Let $\theta : B \longrightarrow B'$ be a ring isomorphism and suppose that*

$$Ext^1_{B'}(V',V\otimes_B B') = Ext^1_{B'}(V\otimes_B B',V') = 0$$

. Then there is a category equivalence

$$F : R\text{-}Mod \rightleftarrows S\text{-}Mod : G$$

with $F(V)=V'$ and $\theta(f)=F(f)$ for all $f\in End({}_RV)=B$.

Moreover , this equivalence is unique up to natural isomorphism .

Proof :

Note that the ring isomorphism $\theta : B \longrightarrow B'$ gives B' a bimodule structure ${}_BB'_{B'}$ and

$$Hom_B(B',-) : B\text{-}Mod \rightleftarrows B'\text{-}Mod : (B'\otimes_{B'} -)$$

defines a category equivalence . By proposition 6.1.4 , there exists an equivalence

$$F' : R\text{-}Mod \rightleftarrows S\text{-}Mod : G'$$

such that $F'(V\otimes_B B')\cong V'$ and there is a natural transformation

$$\eta : V'\otimes_{B'} Hom_B(B',Hom_R(V,-)) \longrightarrow F'$$

which restricted to $\mathcal{T}_{{}_RV}$ is an isomorphism .

In particular , since $V\in\mathcal{T}_{{}_RV}$, there is an isomorphism

$$\eta_V : V'\otimes_{B'} Hom_B(B',Hom_R(V,V)) \cong F'(V)$$

and so if $f \in End({}_RV) = B$, we have that

$$\eta_V \circ (Id_{V'} Hom_B(B', Hom_R(V, f))) = F'(f) \circ \eta_V \quad .$$

$$\begin{tikzcd}
F'V \arrow[rr, "F'(f)"] & & F'V \\
V' \otimes_{B'} Hom_B(B', Hom_R(V,V)) \arrow[u, "\eta_V"] \arrow[rr, "Id_{V'} \otimes Hom_B(B', Hom_R(V,f))"] & & V' \otimes_{B'} Hom_B(B', Hom_R(V,V)) \arrow[u, "\eta_V"']
\end{tikzcd}$$

Note that $B = Hom_R(V, V)$ and

$$e_1 : Hom_B({}_BB'_{B'}, {}_BB'_{B'}) \cong {}_{B'}B'_{B'} \quad [g \mapsto g(1)]$$

. Now we define the S-isomorphism

$$\psi : V' \otimes_{B'} Hom_B(B', Hom_R(V, V)) \to V'$$

$$v' \otimes h \mapsto v' \cdot \theta(h(1))$$

and we claim that , for any $f \in End({}_RV) = B$, the following diagram is commutative .

$$\begin{tikzcd}
V' \otimes_{B'} Hom_B(B', Hom_R(V,V)) \arrow[rr, "Id_{V'} \otimes Hom_B(B', Hom_R(V,f))"] \arrow[d, "\psi"'] & & V' \otimes_{B'} Hom_B(B', Hom_R(V,V)) \arrow[d, "\psi"] \\
V' \arrow[rr, "\theta(f)"] & & V'
\end{tikzcd}$$

To see this , let $v' \otimes h \in V' \otimes_{B'} Hom_B(B', Hom_R(V, V))$, we have

$$\begin{aligned}
& \psi \circ (Id_{V'} \otimes Hom_B(B', Hom_R(V, f)))(v' \otimes h) \\
= \; & \psi(v' \otimes Hom_B(B', Hom_R(V, f))(h)) \\
= \; & v' \cdot \theta(f \circ (h(1)))
\end{aligned}$$

and also

$$\begin{aligned}
\theta(f) \circ \psi(v' \otimes h) &= \theta(f)(v'\theta(h(1))) \\
&= \theta(f)(\theta(h(1))(v')) \\
&= [\theta(f) \circ \theta(h(1)](v') \\
&= (\theta[f \circ h(1)])(v') \qquad (\theta \text{ is a ring iso}) \\
&= v' \cdot \theta(f \circ (h(1)))
\end{aligned}$$

. Therefore the diagram is commutative .

So we have that , for any $f \in End(_RV) = B$,

$$\theta(f) = \psi \circ \eta_V^{-1} \circ F'(f) \circ \eta_V \circ \psi^{-1} \quad .$$

Define now the functor $F : R\text{-}Mod \longrightarrow S\text{-}Mod$ in the following way : for $M, N \in R\text{-}Mod$,

(i) If $M \neq V$, then $F(M) = F'(M)$ and $F(V) = V'$.

(ii) If $g : M \longrightarrow N$ is an homomorphism in $R\text{-}Mod$ then

$$F(g) = \varepsilon_N \circ F'(g) \circ \varepsilon_M^{-1}$$

where

$$\varepsilon_M = \begin{cases} Id_M & \text{if } M \neq V \\ \psi \circ \eta_V^{-1} & \text{if } M = V \end{cases}$$

. Since $F'(V) \cong V'$ and for every $f \in B = End(_RV)$,

$$\begin{aligned}
F(f) &= \varepsilon_V \circ F'(g) \circ \varepsilon_V^{-1} \\
&= \psi \circ \eta_V^{-1} \circ F'(g) \circ \eta_V \circ \psi^{-1} \\
&= \theta(f)
\end{aligned}$$

, thus F is the desired functor .

Suppose there is another equivalence

$$H : R\text{-}Mod \rightleftarrows S\text{-}Mod : K$$

with $H(V) = V'$ and $\theta(f) = H(f)$ for all $f \in End(_RV) = B$. Let $P = {}_RG(S)_S$ and $Q = {}_SF(R)_R$ ($P' = {}_RK(S)_S$ and $Q' = {}_SH(R)_R$) be the progenerators of the equivalence F , G (resp. H , K) such that $F \cong Hom_R(P, -)$ and $H \cong Hom_R(P', -)$.

Then we have the bimodule isomorphisms ,

$$
\begin{aligned}
{}_SQ_R = F(R) &\cong Hom_R(P, R) \cong Hom_R(P, Hom_B(V, V)) \\
&\cong Hom_B(V, Hom_R(P, V)) \quad \text{[AF , 20.7]}
\end{aligned}
$$

. Note that we have the ring isomorphism $\theta : B \longrightarrow B'$ and by proposition 6.1.5 , we have ${}_RV_{B'} \cong V \otimes_B B'_{B'}$ and ${}_SHom_R(P, V)_{B'} \cong Hom_R(P, V)_B \otimes_B B'_{B'}$. Hence

$$
{}_SHom_B(V, Hom_R(P, V))_R \cong_S Hom_{B'}(V \otimes_B B', Hom_R(P, V) \otimes_B B')_R
$$

. And by proposition 6.1.5 , $F \cong Hom_R(P, -)$,

$$
{}_SHom_{B'}(V \otimes_B B', Hom_R(P, V) \otimes_B B')_R \cong_S Hom_{B'}(V \otimes_B B', V')_R
$$

. Similarly we also have that ${}_SQ'_R \cong Hom_{B'}(V \otimes_B B', V')$. So ${}_SQ_R \cong_S Q'_R$ and thus there is a natural isomorphism $Hom_R(P, -) \cong Hom_R(P', -)$.

References

Bibliography of Books

[AF] F. W. Anderson and K. R. Fuller, "Rings and Categories of Modules," 2nd ed., Springer-Verlag, New York/Berlin, 1992.

[Fa] C. Faith, "Algebra II: Ring Theory," Springer-Verlag, New York/Berlin, 1976.

[R] L. Rowen, "Ring Theory ; Vol. II," Academic Press, New York, 1988.

[St] B. Stentrom, "Ring of Quotients," Springer-Verlag, New York/Berlin, 1975.

[W] R. Wisbauer, "Foundations of Module and Ring Theory", Gordon and Breach Science Publishers, 1991.

Bibliography of Articles

[A] G. Azumaya, "Some aspects of Fuller's Theorem," Module Theory, Lect. Notes in Math., Vol. 700 pp. 34-45, Springer-Verlag, New York/Berlin, 1979.

[B] M. Bolla, "Isomorphisms between endomorphism rings of quasiprogenerators", *J. Algebra* **87** (1984), pp. 261-281.

[CbF1] R. R. Colby and K. R. Fuller, "Tilting, cotilting and serially tilted rings," *Comm. Algebra* **18** (1990), pp. 1585-1615.

[CbF2] R. R. Colby and K. R. Fuller, "Tilting and Torsion Theory Counter Equivalences," *Comm. Algebra* **23** (1995), pp. 4833-4849.

[C1] R. Colpi, "Some remarks on equivalences between catogories of modules," *Comm. Algebra* **18** (1990), pp. 1935-1951.

[C2] R. Colpi, "Tilting modules and $*$-modules," *Comm. Algebra* **21** (1993), pp. 1095-1102.

[C3] R. Colpi, "Equivalences between projective and injective modules and morita duality for Artinian rings," *Comm. Algebra* **21** (1993), pp. 601-606.

[CM] R. Colpi and C. Menini, "On the structure of $*$-modules," *J. Algebra* **158** (1993), pp. 400-419.

[D] G. D'Este, "Some remarks on representable equivalences," *Topics in Algebra*, Banach Center Publications, Vol. 26 (1990), pp. 223-232.

[DM] F. Dischinger and W. Muller, "Left PF is not right PF," *Comm. Algebra* **14** (1986), pp. 1223-1227.

[F] K. R. Fuller, "Density and equivalence," *J. Algebra* **29** (1974), pp. 528-550.

[GG] P. A. Guil Asensio and F. Guil Asensio, "On torsion theories induced by tilting modules," *Comm. Algebra* **20** (1992), pp. 2015-2027.

[HR] D. Happel and C. M. Ringel, "Construction of tilted algebras," Lect. Notes in Math., Vol. 903 pp. 125-144, Springer-Verlag, New York/Berlin, 1981.

[LS] T. M. Lok and K. P. Shum, "Isomorphisms between endomorphism rings of quasiprogenerators," *Algebra Colloq.* 2:2 (1995), pp. 157-166.

[M] K. Morita, "Duality for modules and its applications to the theory of rings with minimum condition," *Sci. Rep. Tokyo Kyoiku Daigaku A* **6** (1958), pp. 83-142.

[MO] C. Menini and A. Orsatti, "Representable equivalences between categories of modules and applications," *Rend. Sem. Mat. Univ. Padova* **82** (1989), pp. 203-231.

[S1] M. Sato, "Fuller's theorem on equivalences," *J. Algebra* **52** (1978), pp. 274-284.

[S2] M. Sato, "On equivalences between module subcategories ," *J. Algebra* **59** (1979), pp. 412-420.

[T1] J. Trlifaj, "On $*$-modules generating the injectives," *Rend. Sem. Mat. Univ. Padova* **88** (1992), pp. 211-220.

[T2] J. Trlifaj, "Every $*$-module is finitely generated," *J. Algebra* **169** (1994), pp. 392-398.

[X] W. Xue, "Characterizations of injective cogenerators and Morita duality via equivalences and dualities," *J. Pure and Applied Algebra* **102** (1995) pp. 103-107.